Operator based robust right coprime factorization and control of nonlinear systems

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Chapter 1

Introduction

1.1 Background

In the modern control engineering, as the requirement of the precise and reliable control increases, effective design and accurate control have been becoming the challenging topics.

Since most of the real dynamic systems are nonlinear systems, the control and design for the nonlinear systems have been developing greatly, however, because of the complex structures and the nonlinear dynamics, the development of the nonlinear systems control and design has not been so fast as the linear systems’. The most predominant approach is still the traditional one of using simple, linear, feedback controllers. However, in many cases of engineering and processes, just using linear controllers cannot satisfy the requirement of the control or operation because the design of linear controllers may lead inaccurate performance, affect the work schedule or increase the cost.

Even many effective methods in the control and design of the linear systems can be similarly applied into the control and design of the nonlinear systems, the results cannot be so perfect as the linear systems’. Therefore,
the search and development of an effective approach for nonlinear systems control and design has always been a challenging research topic because of the existence of nonlinearities and uncertainties. Especially, uncertainties are almost unavoidable in the real systems and they usually affect stability of the whole systems which is dangerous for the systems, therefore, considering the security and stability of the systems, it is necessary to eliminate or reduce the effect produced by uncertainties. One of the promising methods is to guarantee the robust stability of the whole system so that the uncertainties cannot make effect to the stability and security of the whole systems. Because of this advantage, robustness has been one of the most interesting and important topics in the control and design fields so that robust control has been attracting more and more attention from various fields, e.g. linear systems and nonlinear systems, time invariant systems and time varying systems, continuous systems and discrete systems. Therefore, designing the robust controllers is a main concern in the system control and design.

In the last three decades, the robust control on the nonlinear systems have received considerable attention and have been studied by many researchers in different fields, as a result, many results and findings have been obtained (see [1], [10] and [72], [79]). A variety of nonlinear control and design techniques have been proposed, such as feedback control method (FC) (see [11], [17]), linear matrix inequality method (LMI) (see [18], [26]), sliding mode control method (SMC) (see [27], [31]), adaptive control approach (AC) (see [32], [36]), right coprime factorization method (RCF) (see [41], [71]) and so on. Among these techniques, the operator based robust right coprime factorization method has been proved to be an effective method of dealing with the control and design of the nonlinear systems. One reason of using this method is that, there is no necessity of assuming all the states
should be measurable which is required by most of the traditional methods. That is, by using the operator based robust right coprime factorization method, control and design of most cases in the control engineering where not all the measurement of the states are available can be realized. Another reason is that the operator is formulated in the extended linear space associated with Banach space which maintains the feasibility of many effective methods in Banach space while the effect of the usually unstable point of infinity is reduced or deleted. The third reason that this method has been receiving much attention is that the defined bounded input bounded output (BIBO) stability can be guaranteed by the satisfaction of the corresponding Bezout identity. In the following, a brief summary of the development of the operator based robust right coprime factorization method is outlined (see [41], - , [71]).

1.2 Developments of coprime factorization

The coprime factorization method has been used to deal with control and design problems on the nonlinear control systems since 1980s, especially the right coprime factorization. The concept of (left and right) coprime factorizations of nonlinear feedback control systems originated from those of transfer matrices in linear feedback control systems. Moreover, the approach of coprime factorization has supplied us a convenient framework to study the input-output stability properties for nonlinear feedback control systems. [53] shows that coprime factorization exist for the input to state mapping of a continuous time nonlinear systems provided that the smooth feedback stabilization problem be solvable for this system. [66] considers an approach using left coprime factorizations of the plant and controllers under certain differential boundedness assumptions. In [52], the authors consider the equivalence
CHAPTER 1. INTRODUCTION

of the class of all (bounded-input) stabilizing nonlinear pre-and feedback-compensators to a class of possibly unstable feedback controllers, and the necessary and sufficient condition of achieving the stability of the system is obtained. Moreover, in [51], the Youla parameterization for linear systems is generalized into a characterization of the class of stable plant controller pairs which is based on the idea of representing the input-output pairs of the plant and controllers as elements of the kernel representation of the systems. Later, the state-space characterization of Youla parameterization is further researched supposed that all nonlinear stabilizing controllers can be expressed by the detectable kernel representation [67]. The case of a stable system with state-space equations affine in the inputs is considered and this system is expressed as the cascade connection of a lossless system, and a stable, minimum-phase system (inner-outer factorization) where the solution is given in terms of the stabilizing solution of a certain Hamilton-Jacobi equation [64]. In [68], the non-square spectral factors of nonlinear input affine state space control systems in continuous time is considered and it is parameterized in terms of a special class of inner-inner factorization. And the normalized right and left coprime factorizations of a nonlinear system are studied from a state space point of view in [69].

Recently, the robust control and robust stabilization of the nonlinear uncertain system is investigated by Chen and Han wherein the right coprime factorization of a nonlinear plant is considered in a fairly general operator setting [56]. That is, the operator can be linear or nonlinear, finite dimensional or infinite dimensional, and can be either in the frequency domain or in the time domain provided that the formulation is meaningful for the physical problems under investigation. In details, firstly, for a given nonlinear plant operator $P$, which can be right factorized into two parts $N$ and $D^{-1}$
in the form of $P = ND^{-1}$ where $N$ and $D$ are two stable operators, if there exist two stable operators $A$ and $B$, together with $N$ and $D$ to satisfy the Bezout identity $AN + BD = M$ ($M$ is a unimodular operator), then the right factorization of the given nonlinear plant is coprime, and the nonlinear system can be proved to be bounded input bounded output (BIBO) stable even though $P$ is unstable. Secondly, considering the robustness of the right coprime factorization which can be linked into the robust stability of the nonlinear systems is investigated by [56] and [59], respectively. In [56], the sufficient condition of guaranteeing the robust stability of the nonlinear feedback systems is given based on the definition of null set which limited the application of the robust right coprime factorization. Therefore, [59] extends the sufficient condition based on Lipschitz norm which can be applied into wider nonlinear systems.

Since the operator based robust right coprime factorization method is different from the traditional methods with the following advantages. 1). Not all the states need to be measured. 2). The stability of the systems can be guaranteed by the satisfaction of the corresponding Bezout identity, that is, the design and control are simpler and easier to be realized. Moreover, the operator based robust right coprime factorization method has been applied into some real nonlinear systems, e.g. based on robust right coprime factorization approach, fault detection in a thermal process control system with input constraints is considered in [60], and networked nonlinear control for an aluminum plate thermal process with time delays is discussed in [62]. Therefore, this method has been proved effective to deal with the design and control problem in the nonlinear systems.
1.3 Motivations of the dissertation

Since the operator based robust right coprime factorization method is proved to be effective for the control and design of the nonlinear system, then the problem of how to getting the right factors of the perturbed nonlinear plant is a fundamental and critical problem and should be firstly considered. Therefore, in this dissertation, the method of factorizing the given plant is proposed by using isomorphism, then some robust control schemes are proposed to guarantee the stability of the nominal nonlinear system as well as the perturbed nonlinear system. Therefore, the right factorization is exactly proved to be robust right coprime factorization of the given plant since the robustness of the factorization is linked to the robust stability of the nonlinear systems. Meanwhile, the plant output tracking problem is also discussed and desired tracking performance is obtained, respectively.

1.4 Contributions of the dissertation

The fundamental problem—the method of factorizing the given nonlinear plant is considered and solved out in this dissertation which gives a foundation and enriches the operator based robust right coprime factorization theory. Meanwhile, some robust control schemes of robust control for nonlinear systems are proposed by which the stability of the nominal nonlinear systems and the perturbed nonlinear systems can be simply guaranteed.

The factorization method and control schemes are proposed based on the operator theory which is formulated in the extended linear space associated with Banach space. The extended linear space associated with Banach space is more suitable for system control theory and engineering concerning with the causality, well-posedness, stability, and robustness of the internal signals.
1.4. CONTRIBUTIONS

for the nonlinear systems. That is because, all the signals in real dynamic systems are time-limited, and most of the existing techniques can be carried over from Banach space to the extended linear space. Moreover, the introductions of bounded input bounded output stability and the generalized Lipschitz operator lead the modeling and control of the nonlinear systems simpler and more precise.

Generally, the robust control is necessary and critical for the control and design of the nonlinear systems since the uncertainties usually exist in the real systems which often affect the stability and safety of the nonlinear systems. Robust control is to design a controller to guarantee the stability of the nominal nonlinear systems as well as the perturbed nonlinear systems. There are many corresponding techniques which are proposed to deal with the robustness of the nonlinear systems, e.g. linear matrix inequality, sliding mode control, adaptive control, among these techniques, the operator based robust right coprime factorization method is effective to deal with the stabilization and robustness of the nonlinear systems.

Different from the general nonlinear control methods, in the control schemes by using operator based robust right coprime factorization method, two stable controllers are designed to satisfy the corresponding Bezout identity for the nominal nonlinear system and the perturbed nonlinear system respectively, which also means that the stability of the nominal nonlinear system as well as that of the perturbed nonlinear system are both guaranteed. That is, the robust control scheme is simple and easy to be realized.

The dissertation is mainly devoted to consider the robust right coprime factorization of the given nonlinear plant and robust control of the nonlinear system. That is, the factorization of the given nonlinear plant is realized by using isomorphism which is a basic concept of mathematics, that is, the
isomorphic subspace of the input space is constructed, then based on the characteristics of isomorphism, the right factorization of the given nonlinear plant can be obtained. This provides the foundation of the operator theory. Then, some robust control schemes based on the obtained factors are designed to guarantee the stability of the nonlinear system which means that the right factorization of the given nonlinear plant is a robust right coprime factorization. Meanwhile, the plant output tracking problems are also discussed and some sufficient conditions are proposed.

In summary, this dissertation considers the robust right coprime factorization and robust control on the operator based nonlinear systems which enriches the control and analysis of the uncertain nonlinear systems.

1.5 Organization of the dissertation

This dissertation is organized as follows.

In Chapter 2, the mathematical preliminaries that serve the foundation of the research consisting of definitions of important spaces and operators are introduced. Firstly, the extended linear space associated with Banach space and generalized Lipschitz operator which is defined on the extended linear space are introduced. Secondly, some basic definitions and results corresponding with robust right coprime factorization such as the right coprime factorization, universal condition and robust conditions are introduced in the setting of operator theory. Thirdly, the definition of isomorphism and its characteristics are outlined which finds the basis and tool of realizing robust right coprime factorization of the given nonlinear plant.

In Chapter 3, firstly, the fundamental problem—the realization of right coprime factorization is considered and the design scheme of the controllers is proposed. That is, the right factorization of the given plant is realized
by using isomorphism wherein the isomorphic subspace of the input space is constructed, then the existence of the controllers is discussed and the existence domains of the controllers are given. Therefore, the right factorization is really a right coprime factorization of the given nonlinear plant.

Secondly, the system design problem is discussed for the nonlinear systems with perturbations. That is, after factorizing the given nonlinear plant, a design scheme of the two stable controllers is proposed, that is, the two stable controllers can be chosen from the given existence domains, together with the obtained factors by using isomorphism based factorization method, the robust conditions and the universal condition are both satisfied. Therefore, the robust stability of the nonlinear systems are guaranteed which implies that the right coprime factorization is exactly to be robust right coprime factorization of the given nonlinear plant since the robustness of the right coprime factorization is linked to the robust stability of the nonlinear system. Moreover, by the proposed design scheme, the plant output tracking property can be realized.

In Chapter 4, for the general cases of constructing the isomorphic subspace of the input space, a sufficient condition is proposed using Lipschitz norm described conditions to logically get the isomorphic subspace of the input space. For the nonlinear systems with perturbations, two stable controllers are designed, together with the factors obtained using isomorphism based factorization method, to satisfy the corresponding Bezout identity for the nominal nonlinear systems and the perturbed nonlinear systems, which implies that the stability of the normal nonlinear system as well as the perturbed nonlinear system is guaranteed. Meanwhile, the plant output can be guaranteed to track to the reference input by the designed controllers.

Further, robust control for the nonlinear feedback systems with unknown
perturbations is considered and a robust control scheme is proposed to simplify the satisfaction of the Bezout identities for the nominal nonlinear systems and the perturbed nonlinear systems. Namely, the stability of the nominal nonlinear system and the perturbed nonlinear system can be simply guaranteed by the proposed design schemes. Meanwhile, the plant output can be guaranteed to asymptotically track to the reference input.

In Chapter 5, the proposed factorization method and robust control schemes are summarized. That is, by the proposed methods, the robust right coprime factorization of the perturbed nonlinear plant is realized and the desired plant output tracking property is also guaranteed.
Chapter 2
Mathematical preliminaries and problem statement

2.1 Introduction

The mathematical preliminaries and problems statement are provided in this chapter to serve the theoretical basis for the research and the following chapters in this dissertation.

In Section 2.2, firstly, the definitions of spaces such as normed space, Banach space, extended linear space associated with Banach space are introduced [57]. Secondly, the definition of operator and some important operators are given which consist of linear and nonlinear operator, invertible operator, stable operator, unimodular operator, Lipschitz operator and generalized Lipschitz operator. Thirdly, the relationship between generalized Lipschitz operator and causality is discussed. Meanwhile, the definitions of bounded input bounded output (BIBO) stability and well-posedness are provided.

In Section 2.3, right factorization, right coprime factorization are firstly provided in the operator theory. Then, a sufficient condition [57] is given to show the relationship between the coprimeness and stability of nonlinear feed-
back systems, based on this relationship, a *universal condition* is proposed to guarantee the coprimeness of the factorization as well as the perfect tracking property. Thirdly, a sufficient robust condition [59] is provided and the relationship between the robust stability of the nonlinear feedback systems and the robust right coprime factorization of perturbed plant is discussed.

In Section 2.4, the definition of isomorphism and its corresponding characteristics such as injective, surjective and bijective properties are provided that serve the technical basis and tool for the realization of the factorization of the given nonlinear plant which is one of the contribution of this dissertation.

In Section 2.5, the main problems which are discussed in this dissertation are introduced. That is, the factorization of the given nonlinear plant and the robust control for the nonlinear system. Meanwhile, the desired plant output tracking properties can be guaranteed by the proposed design schemes.

### 2.2 Mathematical preliminaries

In this section, some basic definitions and notations of spaces, normed linear space, Banach space and extended linear space associated with Banach space are provided, meanwhile, some important results are outlined.

#### 2.2.1 Extended linear space associated with Banach space

In mathematics, many kinds of spaces have been introduced and used in the analysis and calculation. Two basic spaces are linear spaces (also called vector spaces) and topological spaces, where, linear spaces are of algebraic nature, and topological spaces are of analytic nature. Except special statement, this dissertation is based on the linear spaces.
Normed linear space:

Consider a space \( X \) which is said to be a vector space if it is closed under addition and scalar multiplication. The space \( X \) is said to be normed if each element \( x \in X \) is endowed with norm \( \| \cdot \|_X \), such that the following three properties are fulfilled,

1) \( \| x \|_X \) is real, positive number and is different from zero unless \( x \) is identically zero.

2) \( \| ax \|_X = |a| \| x \|_X \).

3) \( \| x_1 + x_2 \|_X \leq \| x_1 \|_X + \| x_2 \|_X \).

Banach space:

Banach spaces are defined as a complete normed vector spaces. This means that a Banach space \( X_B \) is a vector space \( X \) over the real or complex numbers with a norm \( \| \cdot \| \) such that every Cauchy sequence (with respect to the metric \( d(x, y) = \| x - y \|_X \) in \( X \) has a limit in \( X \). As for general vector spaces, a Banach space over the real numbers is called a real Banach space, and a Banach space over the complex numbers is called a complex Banach space.

Extended linear space:

Let \( Z \) be the family of real-valued measurable functions defined on \([0, \infty)\), which is a linear space. For each constant \( T \in [0, \infty) \), let \( P_T \) be the projection operator mapping from \( Z \) to another linear space, \( Z_T \), of measurable functions such that

\[
    f_T(t) := P_T(f)(t) = \begin{cases} f(t), & t \leq T \\ 0, & t > T \end{cases}
\] (2.1)
where, \( f_T(t) \in \mathbb{Z}_T \) is called the truncation of \( f(t) \) with respect to \( T \). Then, for any given Banach space \( X_B \) of measurable functions, set

\[ X^e = \{ f \in \mathbb{Z} : \| f_T \|_{X_B} < \infty, \text{ for all } T < \infty \}. \tag{2.2} \]

Obviously, \( X^e \) is a linear subspace of \( \mathbb{Z} \). The space so defined is called the extended linear space associated with Banach space \( X_B \).

The extended linear space is noted to be not complete in norm in general, and hence it is not a Banach space, but it is determined by a relative Banach space. One reason of choosing extended linear space is that all the control signals are finite time-duration in real systems; The other is that many useful techniques and existing results can be carried over from the standard Banach space \( X_B \) to the extended linear space \( X^e \) if the norm is suitably defined.

### 2.2.2 Generalized Lipschitz operator

Let \( X \) and \( Y \) be linear spaces over the field of real numbers, and let \( X_s \) and \( Y_s \) be normed linear subspaces, called the stable subspaces of \( X \) and \( Y \), respectively, defined suitably by two normed linear spaces under certain norm \( X_s = \{ x \in X : \| x \| < \infty \} \) and \( Y_s = \{ y \in Y : \| y \| < \infty \} \).

**Operator:**

An operator \( Q : X \rightarrow Y \) is a mapping defined from input space \( X \) to the output space \( Y \). The operator \( Q \) can be described as shown in Figure 2.1 and it can also be expressed in the mathematical form as \( y(t) = Q(u)(t) \) where \( u(t) \) is the element of \( X \) and \( y(t) \) is the element of \( Y \).

**Linear and nonlinear operator:**

Consider the operator \( Q : X \rightarrow Y \), where \( \mathcal{D}(Q) \) and \( \mathcal{R}(Q) \) is respectively denoted to be the domain and range of \( Q \). If the operator \( Q : \mathcal{D}(Q) \rightarrow Y \)
satisfies Addition Rule and Multiplication Rule

\[ Q : ax_1 + bx_2 \rightarrow aQ(x_1) + bQ(x_2) \]

for \( \forall x_1, x_2 \in D(Q) \) and \( \forall a, b \in C \) (\( C \) denotes the complex set), then \( Q \) is said to be linear, otherwise, it is said to be nonlinear. Since linearity is a special case of nonlinearity, in what follows ‘nonlinear’ will always mean ‘not necessarily linear’ unless otherwise indicated.

**Bounded input bounded output (BIBO) stability:**

Let \( Q \) be a nonlinear operator with its domain \( D(Q) \subseteq X \) and range \( R(Q) \subseteq Y \). If \( Q(X) \subseteq Y \), \( Q \) is said to be input output stable. If \( Q \) maps all input functions from \( X_s \) into the output space \( Y_s \), that is \( Q(X_s) \subseteq Y_s \), then operator \( Q \) is said to be bounded input bounded output (BIBO) stable or simply, stable. Otherwise, if \( Q \) maps some inputs from \( X_s \) to the set \( Y \setminus Y_s \) (which means that the output of the operator \( Q \) does not belong to the set of \( Y_s \) but belongs to \( Y \)), then \( Q \) is said to be unstable. For any stable operators defined here and later in this dissertation, they always mean BIBO stable.

**Invertible:**

An operator \( Q \) is said to be invertible if there exists an operator \( P \) such that

\[ Q \ast P = P \ast Q = I. \] (2.3)
$P$ is called the inverse of $Q$ and is denoted by $Q^{-1}$, where, $I$ is identity operator and ‘$*$’ is denoted to be the operation defined in the operator theory which can be simply presented as $Q * P$ (or simply $Q(P(\cdot))$ or $QP$).

**Unimodular operator:**

Let $S(X,Y)$ be the set of stable operators mapping from $X$ to $Y$. Then, $S(X,Y)$ contains a subset defined by

$$\mu(X,Y) = \{M : M \in S(X,Y), M \text{ is invertible with } M^{-1} \in S(Y,X)\}. \quad (2.4)$$

Elements of $\mu(X,Y)$ are called unimodular operators.

**Lipschitz operator:**

For any subset $D_B \subseteq X_B$, let $\mathbb{F}(D_B,Y_B)$ be the family of nonlinear operators $Q$ such that $D(Q) = D_B$ and $R(Q) \subseteq Y_B$. Introduce a (semi)-norm into (a subset of) $\mathbb{F}(D_B,Y_B)$ by

$$\|Q\| := \sup_{\frac{x, \hat{x} \in D_B}{x \neq \hat{x}}} \frac{\|Q(x) - Q(\hat{x})\|_{Y_B}}{\|x - \hat{x}\|_{X_B}}$$

if it is finite. In general, it is a semi-norm in the sense that $\|Q\|_{D_B} = 0$ does not necessarily imply $Q = 0$. In fact, it can be easily seen that $\|Q\|_{D_B} = 0$ if $Q$ is a constant operator (need not to be zero) that maps all elements from $D_B$ to the same element in $Y_B$.

Let $\text{Lip}(D_B,Y_B)$ be the subset of $\mathbb{F}(D_B,Y_B)$ with its all elements $Q$ satisfying $\|Q\|_{D_B} < \infty$. Each $Q \in \text{Lip}(D_B,Y_B)$ is called a Lipschitz operator mapping from $D_B$ to $Y_B$, and the number $\|Q\|_{D_B}$ is called the Lipschitz semi-norm of the operator $Q$ on $D_B$.

It is evident that a Lipschitz operator is both bounded and continuous on its domain. Next, generalized Lipschitz operator is introduced, which is defined on extended linear space.
Generalized Lipschitz operator:

Let $X^e$ and $Y^e$ be extended linear spaces associated respectively with two given Banach spaces $X_B$ and $Y_B$ of measurable functions defined on the time domain $[0, \infty)$, and let $D^e$ be a subset of $X^e$. A nonlinear operator $Q : D^e \rightarrow Y^e$ is called a generalized Lipschitz operator on $D^e$ if there exists a constant $L$ such that

$$
\| [Q(x)]_T - [Q(\tilde{x})]_T \|_{Y^e} \leq L \| x_T - \tilde{x}_T \|_{X^e}
$$

(2.5)

for all $x, \tilde{x} \in D^e$ and for all $T \in [0, \infty)$. Note that the least such constant $L$ is given by the norm of $Q$ with

$$
\| Q \|_{Lip} := \| Q(x_0) \|_{Y^e} + \| Q \|_{Y^e}
$$

$$
= \| Q(x_0) \|_{Y^e} + \sup_{T \in [0, \infty)} \sup_{x, \tilde{x} \in D^e \atop x_T \neq \tilde{x}_T} \frac{\| [Q(x)]_T - [Q(\tilde{x})]_T \|_{Y^e}}{\| x_T - \tilde{x}_T \|_{X^e}}
$$

(2.6)

for any fixed $x_0 \in D^e$.

Based on (2.6), it follows immediately that for any $T \in [0, \infty)$

$$
\| [Q(x)]_T - [Q(\tilde{x})]_T \|_{Y^e} \leq \| Q \|_{Lip} \| x_T - \tilde{x}_T \|_{X^e}
$$

$$
\leq \| Q \|_{Lip} \| x_T - \tilde{x}_T \|_{X^e}.
$$

(2.7)

**Lemma 2.1** Let $X^e$ and $Y^e$ be extended linear spaces associated respectively with two given Banach spaces $X_B$ and $Y_B$, respectively, and let $D^e$ be a subset of $X^e$. The following family of Lipschitz operators is a Banach space:

$$
Lip(D^e, Y^e) = \left\{ Q : D^e \rightarrow Y^e \mid \| Q \|_{Lip} < \infty \text{ on } D^e \right\}.
$$

(2.8)

**Proof.** The proof is given in Appendix A.1 [57].
It should be remarked that the family of standard Lipschitz operator and generalized Lipschitz operator are not comparable since they have different domains and ranges. The generalized Lipschitz operator has been proved more useful than standard Lipschitz operator for nonlinear system control and engineering in the considerations of stability, robustness, uniqueness of internal control signals. For any operators defined throughout the paper, they are always assumed to be generalized Lipschitz operators. For simplicity, Lipschitz operator always means the one defined in generalized case in this dissertation. In this dissertation, \( \text{Lip}(D^e) = \text{Lip}(D^e, D^e) \).

### 2.2.3 The relationship between causality and generalized Lipschitz operator

In the following, the relationship between causality and generalized Lipschitz operator is introduced, which is important in the operator theory.

**Causality:**

Let \( X^e \) be the extended linear space associated with a given Banach space \( X_B \), and let \( Q : X^e \to X^e \) be a nonlinear operator describing a nonlinear control system. Then, \( Q \) is said to be causal if and only if

\[
P_TQP_T = P_TQ
\]

for all \( T \in [0, \infty) \), where \( P_T \) is the projection operator.

The physical meaning behind the definition of causality may be understood as follows. If the system outputs depend only on the present and past values of the corresponding system inputs, then we have \( QP_T(u) = Q(u) \) for all input signals \( u \) in the domain of \( Q \), so that \( P_TQP_T = P_TQ \). Conversely, if \( P_TQP_T = P_TQ \) for all \( T \in [0, \infty) \), then we have \( P_TQ(I - P_T)(u) = 0 \) for all input \( u \) in the domain of \( Q \), which implies that any future value of
2.3. OPERATOR BASED RRCF

A system input, \((I - P_T)(u)\), does not affect the present and past values of the corresponding system output given by \(P_TQ(\cdot)\), or in other words, system outputs depend only on the present and past values of the corresponding system inputs.

**Lemma 2.2** A nonlinear operator \(Q : X^e \rightarrow X^e\) is causal if and only if for any \(x, y \in X^e\) and \(T \in [0, \infty)\), \(x_T = y_T\) implies \([Q(x)]_T = [Q(y)]_T\).

**Proof.** The proof is given in Appendix A.2 [57].

**Lemma 2.3** If \(Q : X^e \rightarrow X^e\) is a generalized Lipschitz operator, then \(Q\) is causal.

**Proof.** The proof is given in Appendix A.3 [57].

Note that a nonlinear operator may produce non-unique outputs from an input, particularly for a set-valued mapping. But in real systems, the internal signals in the system are required to be unique. Then **Lemma 2.3** and the following **Lemma 2.4** imply that the uniqueness requirement can be guaranteed by introducing the generalized Lipschitz operator.

**Lemma 2.4** A nonlinear generalized Lipschitz operator produces a unique output from an input in the sense that if the input \(x\) and output \(y\) are related by a generalized Lipschitz operator \(Q\) such that \(y = Q(x)\), then \(x_T = \tilde{x}_T\) implies that \(y_T = \tilde{y}_T\) for all \(T \in [0, \infty)\).

2.3 Operator based robust right coprime factorization

A nominal operator based nonlinear feedback control system is shown in Figure 2.2, where \(U\) and \(Y\) are used to denote the input space and output space of a given plant operator \(P\), respectively, i.e., \(P : U \rightarrow Y\).
2.3.1 Operator based right coprime factorization

Right factorization:

The given plant operator $P : U \to Y$ is said to have a right factorization, if there exist a linear space $W$ and two stable operators $D : W \to U$ and $N : W \to Y$ such that $P = ND^{-1}$ where $D$ is invertible. Such a factorization of $P$ is denoted by $(N, D)$ and $W$ is called a quasi-state space of $P$.

Right coprime factorization:

Let $(N, D)$ be a right factorization of $P$. The factorization is said to be coprime, or $P$ is said to have a right coprime factorization, if there exist two stable operators $A : Y \to U$ and $B : U \to U$, satisfying the Bezout identity

$$AN + BD = M, \text{ for some } M \in \mu(W, U) \tag{2.10}$$

where, $B$ is invertible. Usually, $P$ is unstable and $(N, D, A, B)$ are to be determined which is called to be the system design problem.
2.3. OPERATOR BASED RRCF

It’s worth mentioning that the initial state should also be considered, that is, \( AN(w_0, t_0) + BD(w_0, t_0) = M(w_0, t_0) \) should be satisfied.

In most researches, the researchers choose \( W = U \) briefly, which means that the unimodular operator \( M = I \), where \( I \) is the identity operator. In this dissertation, arbitrary unimodular operator \( M \) is chosen.

2.3.2 Universal condition

Well-posedness:

The feedback control system shown in Figure 2.2 is said to be well-posed, if for every input signal \( r \in U \), all signals in the system (i.e., \( e, u, w, b \) and \( y \)) are uniquely determined.

Overall stable:

The feedback control system shown in Figure 2.2 is said to be overall stable, if \( r \in U_s \), implies that \( u \in U_s, y \in Y_s, w \in W_s, e \in U_s \) and \( b \in U_s \).

Lemma 2.5 Assume that the system shown in Figure 2.2 is well-posed. If the system has a right factorization \( P = ND^{-1} \), then the system is overall stable if and only if the operator \( M \) in (2.10) is a unimodular operator.

Proof. The proof is given in Appendix A.4 [57].

This result can be simplified into: if the plant has a right coprime factorization as \( P = ND^{-1} \), that is, the Bezout identity \( AN + BD = M \) is satisfied, where \( M \) is a unimodular operator, then the system is overall stable.

Moreover, based on the satisfaction of the Bezout identity (2.10), the following equivalent system shown in Figure 2.3 can be obtained and the relationship between the plant output and the reference input can be repre-
CHAPTER 2. MATHEMATICAL PRELIMINARIES

Presented as follow,

\[ y(t) = NM^{-1}(r)(t) \]  \hspace{1cm} (2.11)

if the output space is same with the reference input space, and

\[ NM^{-1} = I \]  \hspace{1cm} (2.12)

then the plant output tracks to the reference input. For simplification, the condition (2.12) is named to be universal condition since in this way of the design scheme of controllers \( A \) and \( B \), both the coprime of the factorization and the plant output tracking property can be guaranteed.

\[ \begin{array}{c}
    r \in \mathcal{Y} \\
    M^{-1} \\
    \mathcal{N} \\
    y \in \mathcal{Y}
\end{array} \]

Figure 2.3: Equivalent system

2.3.3 Operator based robust right coprime factorization

Generally speaking, if the corresponding system with uncertainties remains stable, the system is said to have robust stability property. As to the nonlinear feedback control systems with unknown bounded uncertainties, a sufficient condition of guaranteeing the robustness of the right coprime factorization is derived in [56].

Let’s consider the nonlinear feedback control uncertain system shown in Figure 2.4, where, the nominal plant and real plant is \( P \) and \( \tilde{P} = P + \Delta P \), respectively. The right factorization of the nominal plant \( P \) and the real plant \( \tilde{P} \) are

\[ P = ND^{-1} \]  \hspace{1cm} (2.13)
and

\[ \hat{P} = P + \Delta P = (N + \Delta N)D^{-1} \] (2.14)

where \( N \) and \( D \) are stable operators, \( \Delta N \) is unknown but bounded. Based on the definition of null set, [56] obtained the following sufficient condition of guaranteeing the robust stability of the nonlinear feedback control systems.

\[ A(N + \Delta N) = AN \] (2.15)

under the condition of satisfaction of \( \mathcal{R}(\Delta N) \subseteq \mathcal{N}(A) \), where, \( \mathcal{N}(A) \) is null set defined by

\[ \mathcal{N}(A) = \{ x : x \in \mathcal{D}(A) \text{ and } A(y + x) = Ay \text{ for all } y \in \mathcal{D}(A) \} \] (2.16)

The system is stable because of the fact that

\[ A(N + \Delta N) + BD = AN + BD = M. \] (2.17)

That is, with the uncertainties, the right factorization of the perturbed plant is still coprime.
However, the condition is restrictive so that it cannot be widely applied into the real systems because there seldom exists the case of (2.15). Therefore, in order to improve and extend the condition, a wider condition based on Lipschitz norm is proposed in [59].

**Lemma 2.6** Let $D_e$ be a linear subspace of the extended linear space $U^e$ associated with a given Banach space $U_B$, and let $(A(N + \Delta N) - AN)M^{-1} \in \text{Lip}(D^e)$. Let the Bezout identity of the nominal plant and the exact plant be $AN + BD = M \in \mu(W, U)$, $A(N + \Delta N) + BD = \tilde{M}$, respectively. If

$$
\|(A(N + \Delta N) - AN)M^{-1}\| < 1
$$

(2.18)

then the system shown in Figure 2.4 is stable.

**Proof.** The proof is given in Appendix A.5 [59].

It noted that by satisfying the condition (2.18), $\tilde{M}$ is guaranteed to be a unimodular operator which implies that the establishment of the Bezout identity $A(N + \Delta N) + BD = \tilde{M}$. The Bezout identity is named to be perturbed Bezout identity associated with the perturbed nonlinear system (Uncertainties exist in the nonlinear system) comparing with the nominal Bezout identity corresponding with the nominal nonlinear system (There are no uncertainties in the system).

### 2.4 Definitions and properties of isomorphism

In this dissertation, the basic concept of mathematics: isomorphism is introduced to realize the factorization of the given plant. Isomorphism is an interesting mapping which is popular with graph theory, topology theory, abstract algebra and mathematical analysis. The basic characteristics of isomorphism are given in this section which serves as the technical basis for the
2.4. ISOMORPHISM

research in this dissertation.

2.4.1 Injective mapping

In mathematics, an injective mapping is a mapping that preserves distinctness: it never maps distinct elements of its domain to the same element of its codomain. In other words, every element of the mapping’s codomain is mapped to at most one element of its domain. In the language of mathematics, the definition of injective mapping is defined as follows, where \( \phi \) is a mapping defined in a set \( X \).

The mapping \( \phi \) is injective if \( \forall a, b \in X \), if \( \phi(a) = \phi(b) \), then \( a = b \). Equivalently, if \( a \neq b \), then \( \phi(a) \neq \phi(b) \).

Two examples of injective mapping are shown in Figure 2.5.

![Figure 2.5: Examples of injection](image)

2.4.2 Surjective mapping

A mapping \( \phi \) which takes elements of a set \( X \) and turns them into elements of another set \( Y \) is surjective (or onto) if each element of \( Y \) can be obtained
by applying $\phi$ to some elements of $X$. (There might be multiple elements of $X$ that are turned into the same element of $Y$ by applying $\phi$). A surjective mapping is a mapping whose image is equal to its codomain. In mathematical language, it is as follows, where $X$ and $Y$ denotes the domain and codomain of the mapping, respectively.

$$\phi : X \rightarrow Y, \text{ if } \phi(X) = Y, \text{ then } \phi \text{ is surjective}, \text{ i.e. } \forall y \in Y, \exists x \in X, \text{ such that } \phi(x) = y.$$  

The examples of surjective mapping are shown in Figure 2.6.

![Figure 2.6: Examples of surjection](image)

**2.4.3 Bijective mapping**

From Figure 2.5 and Figure 2.6, we can find that there is one same mapping which is injective and is also surjective. Such a mapping is a bijective mapping.

In mathematics, a *bijection*, or a *bijective mapping*, is a mapping $\phi$ from a set $X$ to a set $Y$ with the property that, for every $y \in Y$, there is exactly one $x \in X$ such that $\phi(x) = y$. It follows from this definition that no unmapped element exists in either $X$ or $Y$. 
2.4. ISOMORPHISM

The example of bijective mapping shown in Figure 2.7 is composed of an injection (left) and a surjection (right).

![Figure 2.7: Examples of bijection](image)

2.4.4 Isomorphism

An isomorphism is a mapping from one algebraic structure to another of the same type that preserves some relevant structures; i.e. properties like identity elements, inverse elements, and binary operations. In mathematics, an isomorphism $\phi : X \rightarrow Y$ is defined such that

$$\phi(x \circ y) = \phi(x) \ast \phi(y) \quad \forall x, y \in X$$

where “$\circ$” is the operation on $X$ and “$\ast$” is the operation on $Y$.

In this dissertation, the operations “$\circ$” and “$\ast$” are considered to be the following quasi-inner product and the operation defined in the operator theory (e.g. The operation between $N$ and $D^{-1}$ in $P = ND^{-1}$ means that the output of operator $D^{-1}$ is the input of operator $N$), respectively.
Quasi-inner product

If $\forall w, u \in X$, the following relationship is satisfied:

$$w \circ u = -\frac{1}{\alpha} u^{\gamma}(t) \left( \int \frac{\alpha w(\tau) - u^{\gamma}(\tau)}{u^{\gamma}(\tau)} d\tau - \beta \right)$$

(2.20)

where $\alpha, \beta \in \mathbb{R}(\alpha \neq 0)$ ($\mathbb{R}$ denotes the set of real numbers) and $\gamma \in \mathbb{N}^+$ ($\mathbb{N}^+$ denotes the set of natural numbers), then the operation “$\circ$” is called the quasi-inner product defined in the space $X$.

2.5 Problem statement

The right coprime factorization method has been applied into the analysis, stabilization, design and control of the nonlinear systems. Most of the existing results are obtained based on the assumption of the acquirement of the right factors, however, in most cases, for a given plant, the factors are unknown which prevents the further research of using robust right coprime factorization. In order to solve this fundamental and critical problem, the problem of factorizing the given plant is proposed and the factorization of the given plant is realized by using isomorphism.

Moreover, the system design for the nonlinear systems with perturbations is considered wherein the existence of the controllers is discussed and the existence domains of the controllers are obtained. By them, the robust conditions and the universal condition are satisfied, that is, the robust stability of the nonlinear feedback control systems is guaranteed while the plant output tracks to the reference input.

Based on Lipschitz norm described conditions, the isomorphism based factorization method is extended to general cases, where the isomorphic subspace of the input space is logically constructed. A quantitative robust control scheme is proposed to guarantee the stability of the nominal nonlinear
systems as well as that of the perturbed nonlinear systems, meanwhile, the plant output tracks to the reference input.

On the nonlinear systems with unknown perturbations, robust stabilizing controllers are designed to simplify the satisfaction of the Bezout identities, which means that the robust control for the nonlinear systems with unknown perturbations can be simply realized. Moreover, the plant output can be guaranteed to asymptotically track to the reference input.

2.6 Conclusion

In this chapter, the mathematical preliminaries including the basic definitions and notations are introduced. Especially, the definitions of extended linear spaces and generalized Lipschitz operators are introduced, which serve a foundation for the dissertation. Moreover, operator based robust right coprime factorization and two sufficient conditions of guaranteeing the stability of the nonlinear system are given in a fairly general operator setting, which provide the theoretical basis for this dissertation. Further, the definition of isomorphism and its characteristics are introduced that serve the technical tools for this research. The concerned problems are also summarized in this chapter.
Chapter 3

Operator based robust right coprime factorization of nonlinear systems

3.1 Introduction

It has been difficult to deal with the nonlinear control system design problems because of the complex structures and the nonlinear dynamics. Even if the effective methods in the linear control systems are similarly applied into the nonlinear control systems, there have not been such perfect results as the linear systems’. Therefore, many researchers have been researching into some methods to cope with the problems existing in the nonlinear control systems. Among the methods, the coprime factorization method has been considered to be effective to study operator based nonlinear control system design problems for the nonlinear feedback control systems, including stability analysis, output tracking design, and fault detection and so on.

Recently, the method of right coprime factorization has been applied into some real cases ([60], -, [62]). The networked nonlinear control for an aluminum plate thermal process [62], nonlinear vibration control of a flexible
arm experimental system with uncertainties [61] and fault detection for an uncertain aluminum plate thermal process control system with input constraints [60] have been studied using the operator based robust right coprime factorization. The right coprime factorization method has been a promising method for the control problem of the nonlinear feedback control systems.

However, there are seldom papers considering the method of factorizing the nonlinear plant, though many researchers have been applying and developing coprime factorization method from various aspects and various fields. Without the forms or value of the factors, there is no way of dealing with the robust stability and tracking problems. Even if the right factors of the given nonlinear plant are known, there should exist two stable operators \((A, B)\) which can be used as the controllers, together with the obtained factors \((N, D)\), to satisfy the Bezout identity. However, in many cases, the design of the controllers \((A, B)\) is difficult which imposes a severe restriction on nonlinear control. Therefore, based on the isomorphism approach, the factorization problem of nonlinear plants is solved and the controllers \((A, B)\) can be simply designed.

Isomorphism is a useful concept or tool in mathematics such as graph theory, topology theory, abstract algebra and mathematical analysis (see [37], - , [40]). Essentially, isomorphism is an interesting map which preserves some properties from one algebraic structure to another one. It supplies a simple method to understand or analyze an unknown structure according to the knowledge of some familiar structures such as Euclidean Space, Banach Space and Hilbert Space. [40] considers the identifiability of the homogeneous systems with specified initial states using the state isomorphism approach. The Mislin’s theorem on group homomorphisms is proved based on ideas of Alperin and the use of Lannes’s T-functor, and the Mislin’s the-
Theorem induce an isomorphism in cohomology in [39]. The author extends the class of groups to include groups of finite virtual cohomological dimension and profinite groups. A duality for near-isomorphism categories of almost completely decomposable groups through a combination of Butler duality and Warfield duality is established in [38]. [37] considers the relative Spring isomorphism from the aspect of the algebraic group over the algebraically closed field. However, most of the above papers are just researches in the field of pure theoretical mathematics. It is well known that applying the mathematic theories and methods into the application of the control problems is an important issue. Therefore, the isomorphism is introduced to construct the isomorphic subspace of the input space so that factors of the right coprime factorization can be obtained. In other words, the right coprime factorization of the nonlinear unstable plant is realized by means of isomorphism approach.

In Section 3.2, the right coprime factorization of a nonlinear unstable plant is considered, where the exact forms or value of the factors are unknown. An isomorphism based right coprime factorization method is proposed to factorize the unstable plant. Then, according to the obtained factors, the design schemes of the two stable controllers are discussed and the existence domains of the controllers are given, which means that the right factorization is coprime and the stability of the nonlinear system is guaranteed. Further, according to the obtained factors, the sufficient conditions of guaranteeing the plant output tracking to the reference input are provided. Finally, a numerical example is given to show the validity of the proposed methods.

In Section 3.3, the system design of the nonlinear systems is considered. In detail, the right factorization of the given nonlinear plant with perturbations is realized by isomorphism, then the existence domains of two stable
controllers are obtained and by the designed stabilizing controllers, the robust conditions are satisfied. Namely, not only the stability of the nominal nonlinear system but also that of the perturbed nonlinear system are guaranteed. As a result, the right factorization is exactly to be a robust right coprime factorization of the perturbed plant. Meanwhile, the designed two robust controllers, together with the obtained factors, satisfy the universal condition, which implies that the plant output tracking to the reference input property is guaranteed. The given numerical example confirms the effectiveness of the proposed method.

In Section 3.4, the factorization method of robust right coprime factorization of the nonlinear plant is summarized.

### 3.2 Isomorphism based right coprime factorization and the output tracking properties

In this section, a method of acquiring the right factorization of the unstable plant is proposed, where the quasi-inner product (2.20) is chosen to be simple with $\gamma = 1$.

**Theorem 3.1** Suppose that the system shown in Figure 2.2 is well-posed, based on the approach of isomorphism, the right factorization of the unstable plant can be realized.

**Proof:** Assume that $P$ has a right factorization $P = ND^{-1}$ as the system shown in Figure 2.2, where $N$, $D$ are stable and unknown, $D^{-1}$ is unstable.

An isomorphic subspace $\tilde{W}$ of the input space $U$ will be constructed. First, a stable operator $S^{-1}$ is supposed to be designed to stabilize the unstable plant $D^{-1}$ which is shown in Figure 3.1.
The equivalent system of the system shown in Figure 3.1 is simply expressed by an operator $\tilde{D}^{-1}$, that is $\tilde{D}^{-1} : U \rightarrow \tilde{W}$ which is shown in Figure 3.2, where the input space and the output space is $U$ and $\tilde{W}$, respectively.

Then there exists an isomorphism $\phi$ between $U$ and $\tilde{W}$, so the following equation is satisfied:

$$\phi(w \circ u) = \phi(w) \ast \phi(u) \quad (3.1)$$

where “$\circ$” is the quasi-inner product defined in space $U$ and “$\ast$” is the operation defined in operator theory, therefore, $w = \phi(u)$ is established in the right side of (3.1), then according to the definition of isomorphism, $w = \phi(u)$ is also established in the left side of equation (3.1), so the following equation
(3.2) is satisfied:

\[ \phi(\phi(u) \circ u) = \phi(\phi(u)) \]  \hspace{1cm} (3.2)

then by the injective property of isomorphism,

\[ \phi(u) \circ u = \phi(u) \]  \hspace{1cm} (3.3)

then we can use (2.20) and an elementary Gronwall’s equality to conclude that

\[ \phi(u) = g(t)u \]  \hspace{1cm} (3.4)

So we can define the following two operators:

\[ \tilde{D}^{-1}(u) = \phi(u) = g(t)u \]
\[ N(w) = g(t)w \]  \hspace{1cm} (3.5)

According to the formalism of the operator \( P = ND^{-1} \), the factor \( D^{-1} \) of the plant \( P \) can also be obtained, then \( S^{-1} \) can be derived. Therefore, based on the approach of isomorphism, the right factorization of the unstable plant can be realized. This completes the proof.

In the above proof, the isomorphism is one special kind which can be named as quasi-isomorphism because of the relationship between \( w \) and \( u \):

\( w = \phi(u) \). \( N(w), \tilde{D}^{-1}(u) \) are in the linear forms of \( w, u \), respectively.

Therefore, the problem of the unstable plant \( P = ND^{-1} \) is transformed into the similar one of the stable plant \( \hat{P} = N\hat{D}^{-1} \), where the operator \( \hat{D}^{-1} \) is stable. However, whether \( \hat{D} \) is stable or not needs to be judged according to the real systems. Consequently, the operator \( \hat{D} \) will be discussed in two parts: stable and unstable.
3.2.1 Case 1: stable factor $\tilde{D}$

Based on the isomorphism factorization method, the obtained factors are of linear forms, so the solutions $A$ and $B$ of the Bezout identity $AN + BD = M$ can be designed in the linear forms for a given unimodular $M$ which is also of linear form. That is, in the linear forms, there necessarily exist two stable operators $A$ and $B$ to satisfy the Bezout identity so that the right factorization is coprime. For simplification, only the condition that the coefficients of all the operators are positive definite is studied in the following proof and other conditions for the coefficients can be similarly discussed.

Figure 3.3: Obtained nonlinear feedback system

Theorem 3.2: Consider the system shown in Figure 3.3, where the operators $N$ and $\tilde{D}^{-1}$ are obtained from the isomorphism factorization method, then for a unimodular $M$, there exist two stable operators $\tilde{A}$ and $\tilde{B}$ to satisfy the Bezout identity, so the right factorization is coprime.

Proof: Since all the operators $N$, $\tilde{D}$, $\tilde{A}$, $\tilde{B}$ and $M$ are of the linear forms, so it is sufficient to only consider their coefficients to satisfy the Bezout identity.
Suppose that the coefficients of the operators $N$, $\tilde{D}$, $\tilde{A}$, $\tilde{B}$ and $M$ be expressed by $n$, $\tilde{d}$, $\tilde{a}$, $\tilde{b}$ and $m$, respectively, where $n$, $\tilde{d}$, $\tilde{a}$, $\tilde{b}$ and $m$ are the continuous functions of $t$. If the Bezout identity is satisfied, then the following equation about coefficients of all the operators is also satisfied:

$$\tilde{a}n + \tilde{b}\tilde{d} = m$$  \hfill (3.6)

then

$$\tilde{a} = (m - \tilde{bd})n^{-1}$$

$$\tilde{b} = (m - \tilde{an})\tilde{d}^{-1}$$  \hfill (3.7)

since the coefficients of all the operators are positive definite, so the existence domains of $\tilde{a}$ and $\tilde{b}$ can be obtained:

$$0 < \tilde{a} < mn^{-1}$$

$$0 < \tilde{b} < m\tilde{d}^{-1}$$  \hfill (3.8)

therefore $\tilde{A}$ and $\tilde{B}$ can be chosen from the following sets:

$$A = \{ \tilde{A}(y) = \tilde{ay} | 0 < \tilde{a} < mn^{-1}, y \in \tilde{Y} \}$$

$$B = \{ \tilde{B}(u) = \tilde{bu} | 0 < \tilde{b} < m\tilde{d}^{-1}, u \in U \}$$  \hfill (3.9)

This completes the proof.

**Theorem 3.3:** Assume that the system shown in Figure 3.3 is well-posed, where the operator $\tilde{D}$ is stable. For a given unimodular $M$ in the linear form, two stable operators $\tilde{A}$, $\tilde{B}$ can be designed (which can be chosen from the sets $A$ and $B$) to satisfy the Bezout identity

$$\tilde{A}N + \tilde{B}\tilde{D} = M \in \mu(\tilde{W}, U)$$  \hfill (3.10)
if the following equation is satisfied:

\[ NM^{-1} = I \]  (3.11)

then the system is overall stable and the output tracks to the reference input.

Proof: The system shown in Figure 3.3 is stable by Lemma 2.5 and can be simplified to be equivalent with the system in Figure 3.4, and:

\[ y = NM^{-1}(r) = I(r) = r \]  (3.12)

so the output tracks to the reference input.

### 3.2.2 Case 2: unstable factor \( \tilde{D} \)

In this section, the operator \( \tilde{D} \) is unstable. However, according to the definition of right coprime factorization, all the operators \( A, B, N \) and \( D \) need to be stable. So the unstable operator \( \tilde{D} \) has to be stabilized in order that the Bezout identity can be satisfied.

**Theorem 3.4** Consider the system with a plant \( \hat{P} = N \hat{D}^{-1} \), where the operator \( \hat{D} \) is unstable. \( \hat{D} \) can be stabilized by \( \hat{S}^{-1} \) which is shown in Figure 3.5, and the equivalent system is supposed to be \( \hat{D}^{-1} \), where \( \hat{D} \) is stable. For a given unimodular \( \hat{M} \), two stable operators \( \hat{A} \) and \( \hat{B} \) can be designed to satisfy the Bezout identity, so the whole system is overall stable.
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Figure 3.5: Stabilization of $\tilde{D}$

Figure 3.6: Nonlinear system after factorization

Figure 3.7: Tracking scheme with tracking compensator
\( \hat{A} \) and \( \hat{B} \) can be chosen from two sets \( \mathfrak{A} \) and \( \mathfrak{B} \) which are similar with \( A \) and \( B \) based on Theorem 3.2.

**Theorem 3.5** For the system in Figure 3.6, based on the Bezout identity the whole system can be simplified into the equivalent system shown in Figure 3.7, where \( L \) is a tracking compensator, and if the following equation is satisfied:

\[
N\hat{M}^{-1}L = I \tag{3.13}
\]

then the output \( y \) can track to the reference input \( r^* \).

In this section, first, a new method of realizing a factorization of the nonlinear unstable plant is proposed in Theorem 3.1. Then, the existence domains of two stable operators \( A \) and \( B \) for satisfying the Bezout identity are discussed in Theorem 3.2. As a result, the stability of the nonlinear feedback system and the output tracking problems are studied, respectively. If the obtained operator \( \hat{D} \) is stable, a universal design scheme is proposed in Theorem 3.3 which guarantees the stability of the nonlinear feedback control system and realizes the output tracking to the reference input. If \( \hat{D} \) is unstable, the sufficient conditions on stability and tracking problems are given in Theorem 3.4 and Theorem 3.5, respectively.

### 3.2.3 Numerical example

In this section, a numerical example is given to show the effectiveness of the proposed method.

First, two spaces \( U \) and \( Y \) are given as follows:

\[
U = C_{[0, \infty)} \quad \quad Y = \{y(t) | y(t) = \beta(t)u^2(t), u(t) \in U \}
\]

where \( C \) is a set of continuous functions which are defined in \([0, \infty)\) and \( \beta(t) \) is also a continuous function.
Let \( \| \cdot \| \) be the sup-norm defined by

\[
\| u \|_\infty = \sup_{t \in [0, \infty)} |u(t)|
\]

and define

\[
U_s = \{ u(t) | u(t) \in U, \| u \|_\infty < \infty \}
\]

Similarly

\[
Y_s = \{ y(t) | y(t) \in Y, \| y \|_\infty < \infty \}
\]

It can be verified that both of the above spaces are linear normed, so they can be used to be the stable subspaces of \( U \) and \( Y \), respectively.

Consider the system in Figure 2.2 in which the input space, output space is \( U, Y \), respectively. The given operator \( P : U \rightarrow Y \) is defined by:

\[
P(u)(t) = (e^{2t} + e^t)u^2(t)
\]  

(3.14)

Next, an isomorphic space \( \tilde{W} \) of \( U \), which is also a subspace of \( U \) is constructed by feedback shown in Figure 3.1. The mapping \( \phi : U \rightarrow \tilde{W} \) is defined to be an isomorphism between two spaces. So according to the characteristics of isomorphism, the following relationship reads:

\[
\phi(w \circ u) = \phi(w) \ast \phi(u)
\]  

(3.15)

where \( w, u \in U \) and “*” is the operation in operator theory, so \( w = \phi(u) \), then:

\[
\phi(\phi(u) \circ u) = \phi(\phi(u))
\]  

(3.16)

and the mapping ‘\circ’ is defined by:

\[
\phi(u)(t) \circ u(t) = \frac{2}{5}u(t)\left( \int_0^t \frac{\frac{5}{2}\phi(u(\tau)) - u(\tau)}{u(\tau)}d\tau - 2 \right)
\]  

(3.17)
then by the injective property of isomorphism:

\[ \phi(u)(t) \circ u(t) = \phi(u)(t) \]

So:

\[ -\frac{2}{5}u(t)\left(\int_0^t \frac{5}{2}\phi(u)(\tau) - u(\tau)\right)d\tau - 2) = \phi(u)(t) \tag{3.18} \]

then

\[ \phi(u)(t) = \frac{2}{5}(1 + e^{-t})u(t) \tag{3.19} \]

So we can define the operators \( N \) and \( \tilde{D}^{-1} \) to be:

\[
N(w)(t) = \frac{2}{5}(1 + e^{-t})w(t) \\
\tilde{D}^{-1}(u)(t) = \frac{2}{5}(1 + e^{-t})u(t) \tag{3.20}
\]

then the operators \( D^{-1}(u)(t) = \frac{5}{2}e^{2t}u^2(t) \) and \( S^{-1}(z)(t) = \left(\frac{4e^{-2t}(e^{-t}+1)}{5(3-2e^{-t})}\right)z(t) \frac{1}{2} \)
can be obtained. And the operators \( N \) and \( \tilde{D}^{-1} \) are both stable.

Last, for the given unimodular operator \( M(w)(t) = \frac{2}{5}(1 + e^{-t})w(t) \), two stable operators \( \tilde{A} \) and \( \tilde{B} \) can be designed from the sets \( A \) and \( B \) according to Theorem 3.2, respectively

\[
\tilde{A}(y)(t) = \frac{1}{2}y(t) \\
\tilde{B}(u)(t) = \frac{2}{25}(1 + e^{-t})^2u(t) \tag{3.21}
\]

where \( A \) and \( B \) are given as follows

\[
A = \{ \tilde{A}(y) = \tilde{a}y|0 < \tilde{a} < 1, y \in \bar{Y} \} \\
B = \{ \tilde{B}(u) = \tilde{b}u|0 < \tilde{b} < \frac{4}{25}(1 + e^{-t})^2, u \in U \} \tag{3.22}
\]
so the following equation is satisfied:

\[(\tilde{A}N + \tilde{B}D)(w)(t) = \tilde{A}[\frac{2}{5}(1 + e^{-t})w(t)] + \tilde{B}[\frac{5e^t}{2e^t + 2}w(t)]\]
\[= \frac{1}{5}(1 + e^{-t})w(t) + \frac{2}{25}(\frac{e^t + 1}{e^t})^2 \frac{5e^t}{2e^t + 2}w(t)\]
\[= \frac{2}{5}(1 + e^{-t})w(t)\]
\[= M(w)(t)\] (3.23)

The tracking condition is also satisfied:

\[y(t) = NM^{-1}(r)(t) = \frac{2}{5}(1 + e^{-t})\frac{5e^t}{2e^t + 2}r(t) = r(t)\] (3.24)

The results are shown in Figures 3.8 - 3.10, and it is obvious that the plant output \(y_1\) tracks to the reference input while reference input, plant output and quasi-state are all stable, where the reference input is: \(r(t) = 1 + 3e^{-t} - e^{-10t}\).

![Figure 3.8: Reference input r](image)

Therefore, from the simulation results, we can find that the effectiveness of the proposed method is confirmed.
Figure 3.9: Plant output $y$

Figure 3.10: Quasi-state $w$
3.3 System design using operator based robust right coprime factorization and isomorphism

3.3.1 Mathematical preliminaries

On the issue of robust control for nonlinear system, besides the sufficient conditions in Lemma 2.6, there are other important conditions of guaranteeing the robust stability of the nonlinear system. In detail, the nonlinear plant is considered to be factorized as \((P + \Delta P) = (N + \Delta N)(D + \Delta D)^{-1}\), where the uncertainties and the effect of the time delay are considered in \(\Delta N\) and \(\Delta D\), respectively.

\[
P + \Delta P
\]

Figure 3.11: The perturbed nonlinear feedback system

Lemma 3.1 Let \(D^e\) be a linear subspace of the extended linear space \(U^e\) associated with a given Banach space \(U_B\), let \(M^{-1}[A(N + \Delta N) - AN + B(D + \Delta D) - BD] \in \text{Lip}(D^e)\). Let the Bezout identity of the nominal plant and the exact plant be \(AN + BD = M \in \mu(W, Y)\), \(A(N + \Delta N) + B(D + \Delta D) = \tilde{M}\),
respectively. If the following condition is satisfied:

\[ \left\| [A(N + \Delta N) - AN + B(D + \Delta D) - BD]M^{-1} \right\| < 1 \]  (3.25)

the system shown in Figure 3.11 is stable, where \( \| \cdot \| \) is the Lipschitz norm.

Proof. The proof is given in Appendix A.6 [62].

It is worth mentioning that

i) If the perturbed plant of the nonlinear feedback system still remains a right coprime factorization under some conditions, then the perturbed plant is said to have a robust right coprime factorization.

ii) Under some sufficient conditions, if the perturbed nonlinear feedback control system still remains overall stable, then the perturbed nonlinear system is said to have the robust stability property.

iii) Moreover, the robust stability of the nonlinear feedback system implies the robust right coprime factorization of the perturbed nonlinear plant.

Generally, the robustness of the right coprime factorization is linked to the robust stability of the nonlinear feedback control system. Therefore, under the above conditions, the perturbed plant can also be said to have a robust right coprime factorization. For simplification, in the following parts of this dissertation, the conditions (3.25) as well as the conditions (2.18) are named to be robust conditions.

Another problem that this paper is concerned with is whether the plant output track to the reference input or not, and on this problem, there are some results in the existing results, especially the universal condition by which the tracking property and stability of the nonlinear feedback control system can be both guaranteed.
3.3.2 Problem statement

There have been some remarkable results on the design and control of the robust right coprime factorization for nonlinear feedback control systems, however, there still exists a fundamental problem which gets less attention, that is, how to factorize the plant so that there necessarily exist two stable controllers to guarantee that the perturbed plant has a robust right coprime factorization. Therefore, in this section, the problem will be discussed for some kinds of nonlinear plants.

Considering the nonlinear feedback system shown in Figure 3.11 where the real plant $P + \Delta P$ and the model plant $P$ are known to have a right factorization, respectively: $P + \Delta P = (N + \Delta N)(D + \Delta D)^{-1}$ and $P = ND^{-1}$, where $N$, $N + \Delta N$, $D$ and $D + \Delta D$ are stable while $\Delta N$, $\Delta D$ are bounded. Maybe there are several forms of right factorization for the nonlinear perturbed plants, but whether they are robust right coprime factorizations or not is unknown, and this is the main issue of this section In the following parts, the robust right coprime factorization of the perturbed plant will be realized by using isomorphism. In detail, the right factorization of the perturbed plant is realized by using isomorphism, then for the obtained two factors, there necessarily exist two stable controllers to guarantee the robust stability of the obtained nonlinear feedback system as well as the plant output tracking to the reference input. Therefore, the right factorization is exactly to be a robust right coprime factorization of the perturbed nonlinear plant while the plant output tracking property is also realized.

3.3.3 Isomorphism based factorization method

Theorem 3.6: Suppose that the system shown in Figure 3.11 is well-posed, the right factorization of the plant can be realized using isomorphism approach
where $\tilde{D} + \Delta \tilde{D} - I$ is designed to be invertible and its inverse is stable.

\[
\begin{array}{cccccc}
  & u & \in & \tilde{U} & e & \in & \tilde{U} \\
  & & \downarrow & & \downarrow & & \downarrow \\
  & S^{-1} & & (D + \Delta D)^{-1} & & w & \in & \tilde{W} \\
  & - & & & & w & \in & \tilde{W} \\
\end{array}
\]

Figure 3.12: Feedback system with $S^{-1}$

\[
\begin{array}{cccccc}
  & (\tilde{D} + \Delta \tilde{D})^{-1} & \downarrow & & & & \Delta N \\
  & \downarrow & & & \downarrow & & (\tilde{D} + \Delta \tilde{D})^{-1} \\
  & \Delta N & & & N & & + \\
\end{array}
\]

Figure 3.13: Two parts obtained by isomorphism

**Proof:** As the problem statement discussed, we consider the method of factorizing the real plant $P + \Delta P$ into two parts $N + \Delta N$ and $(D + \Delta D)^{-1}$.

First, an isomorphic subspace $\tilde{W}$ of $\tilde{U}$ ($\tilde{U} \subseteq U$, $\tilde{W} \subseteq \tilde{U}$, where $\tilde{U}$ is one positive and stabilizable subspace of $U$) will be constructed. A compensator $S^{-1}$ is supposed to be designed as shown in Figure 3.12. Its equivalent system $(\tilde{D} + \Delta \tilde{D})^{-1}$ is shown in Figure 3.13, where input space and output space is $\tilde{U}$ and $\tilde{W}$, respectively.

Since the space $\tilde{W}$ is the isomorphic space of the input space $\tilde{U}$, the isomorphism between $\tilde{U}$ and $\tilde{W}$ can be assumed to be $\phi$, then for $w, u \in \tilde{U}$, the following relationship is satisfied:

$$\phi(w \circ u) = \phi(w) * \phi(u)$$  \hspace{1cm} (3.26)
where “$\circ$” is the quasi-inner product defined in space $\hat{U}$ and “$\ast$” is the operation defined in the operator theory, thus, $w = \phi(u)$ and the following relationship is established:

$$
\phi(\phi(u) \circ u) = \phi(\phi(u))
$$

(3.27)

then by the injective property of isomorphism, (3.28) can be obtained:

$$
\phi(u) \circ u = \phi(u)
$$

(3.28)

The result of the operation between $\phi(u)$ and $u$ is generally assumed to be $\Phi(\phi(u), u)$, then

$$
\Phi(\phi(u), u) = \phi(u)
$$

(3.29)

Then according to an elementary Gronwall’s equality, the solution of (3.29) can be supposed to be $\phi(u) = \Psi(t)u^\gamma$, where $0 < \min\{\Psi(t)\} < \Psi(t) < \max\{\Psi(t)\}$ for $t \in [0, \infty)$, thus the following two operators can be designed:

$$
(N + \Delta N)(w)(t) = \Psi(t)w^\gamma(t) + \Delta(t)w^\gamma(t)
$$

(3.30)

$$
(\tilde{D} + \Delta \tilde{D})^{-1}(w)(t) = \Psi(t)u^\gamma(t) + \Delta(t)u^\gamma(t)
$$

(3.31)

where $\Delta(t)u^\gamma(t)$ denotes the perturbations included in the part $N + \Delta N$ as well as the resulting perturbations in $(\tilde{D} + \Delta \tilde{D})^{-1}$. According to the definition of BIBO stability, the operators $N + \Delta N, (\tilde{D} + \Delta \tilde{D})^{-1}$ and $(\tilde{D} + \Delta \tilde{D})$ are all stable. $(D + \Delta D)^{-1}$ can be obtained by $P + \Delta P = (N + \Delta N)(D + \Delta D)^{-1}$.

Moreover, the existence of the compensator $S^{-1}$ will be discussed as follows. According to the system in Figure 3.12 and its equivalent system $(\tilde{D} + \Delta \tilde{D})^{-1}$, the following two equalities are satisfied:

$$
u = S(D + \Delta D)(w) + I(w), \quad u = (\tilde{D} + \Delta \tilde{D})(w)
$$

(3.32)
thus, \( S(D + \Delta D) = \tilde{D} + \Delta \tilde{D} - I \), since \((\tilde{D} + \Delta \tilde{D} - I)\) is invertible and its inverse is stable, then \( S^{-1} = (D + \Delta D)(\tilde{D} + \Delta \tilde{D} - I)^{-1} \) is also stable. Therefore, by constructing an isomorphic space of the space \( \tilde{U} \), the right factorization of the plant can be realized. This completes the proof.

The right factorization of the perturbed plant has been realized, but whether the right factorization is coprime or not is unknown, which means that we have to check it whether there exist two stable controllers to satisfy a Bezout identity or not. If the two stable controllers exist, then the right factorization is coprime; otherwise, it isn’t. Next, the existences of the stable controllers will be discussed.

### 3.3.4 Existence of two controllers \( \tilde{A} \) and \( \tilde{B} \)

From the existing results, we can see that the controllers \( A \) and \( B \) directly affect the robust stability of the nonlinear feedback control systems and indirectly affect the property of the plant output tracking to the reference input. In detail, if the two stable controllers \( A \) and \( B \) exist and the Bezout identity can be satisfied, then the right factorization is coprime (Lemma 2.5). If the two controllers \( A \) and \( B \) are designed such that the robust conditions can be satisfied, then the robust stability of the nonlinear feedback control systems can be guaranteed. Moreover, if the universal condition can be satisfied by the two designed controllers \( A \) and \( B \), then the robust stability and the plant output tracking property can be both guaranteed by the universal condition. Therefore, the two stable controllers are important for the robust stability and the output tracking properties, then existence of the two controllers \( \tilde{A} \) and \( \tilde{B} \) will be discussed for the obtained nonlinear feedback control systems.

**Theorem 3.7**: Considering the nonlinear feedback control system shown in Figure 3.14, where the system is well-posed, under the universal condition
which guarantees the plant output track to the reference input, the two controllers \( \tilde{A} \) and \( \tilde{B} \) exist and are stable.

**Proof:** Suppose that the two controllers \( \tilde{A} \) and \( \tilde{B} \) exist, according to universal condition, we can get the following condition for the obtained nonlinear feedback control system:

\[
\tilde{A}(N + \Delta N) + \tilde{B}(\tilde{D} + \Delta \tilde{D}) = N + \Delta N
\] (3.33)

then for convenience, we consider the form of controller \( \tilde{A} \) to be

\[
\tilde{A}(y) = \frac{m}{n} y, \quad \text{where } n, m \in \mathbb{N}^+, 0 < m < n, n \neq \infty
\] (3.34)

\( \tilde{A}(y) \) can be found to be stable and \( I - \tilde{A} \) is invertible: \( (I - \tilde{A})^{-1}(e) = \frac{n}{n-m} e \).

Therefore,

\[
\tilde{B}(u) = (I - \tilde{A})(N + \Delta N)(\tilde{D} + \Delta \tilde{D})^{-1}(u)
\] (3.35)

and

\[
\tilde{B}^{-1}(e) = (\tilde{D} + \Delta \tilde{D})(N + \Delta N)^{-1}(I - \tilde{A})^{-1}(e)
\] (3.36)

then according to the definition of BIBO stability, \( \tilde{A}, \tilde{B}, \) and \( \tilde{B}^{-1} \) are all found to be stable.
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3.3.5 Robust stability

Since the existence and stability of the two controllers $\tilde{A}$ and $\tilde{B}$ are proved in the above section, next, the robust stability of the nonlinear feedback control systems will be discussed.

**Theorem 3.8:** Considering the nonlinear feedback control system shown in Figure 3.14, where the controllers $\tilde{A}$ and $\tilde{B}$ are in the forms of (3.34) and (3.35), respectively, then there exists a unimodular operator $M \in \mu(\tilde{W}, \tilde{Y})$ to satisfy the nominal Bezout identity $\tilde{A}N + \tilde{B}\tilde{D} = M$.

**Proof:** Suppose there exists an operator $M$ to satisfy that $M(w) = (\tilde{A}N + \tilde{B}\tilde{D})(w)$, then according to (3.30), (3.31), (3.34) and (3.35), the following equality can be obtained:

$$
M(w)(t) = (\tilde{A}N + \tilde{B}\tilde{D})(w)(t) \\
= \frac{m}{n} N(w)(t) + \frac{n - m}{n} (N + \Delta N)(\tilde{D} + \Delta \tilde{D})^{-1} \tilde{D}(w)(t) \\
= \frac{m}{n} \Psi(t)w^\gamma(t) + \frac{n - m}{n} [\Psi(t) + \Delta(t)][\frac{\Psi(t) + \Delta(t)}{\Psi(t)}]^\gamma w^\gamma(t) \\
= \left[ \frac{m}{n} \Psi(t) + \frac{n - m}{n} \frac{[\Psi(t) + \Delta(t)]^{\gamma + 1}}{\Psi^\gamma(t)} \right] w^\gamma(t) \\
= \Sigma(t)w^\gamma(t)
$$
where \( \Sigma(t) = \frac{m}{n} \Psi(t) + \frac{n-m}{n} \frac{\Psi(t) + \Delta(t)}{\Psi(t)} \). We can find that \( \Sigma(t) \) is stable, thus \( M(w)(t) \) is stable. Moreover \( M^{-1}(w)(t) = \left[ \frac{1}{\Sigma(t)} w(t) \right]^\frac{1}{2} \) is also stable. As a result, the operator \( M \) is a unimodular operator.

**Theorem 3.9**: Considering the nonlinear feedback control system shown in Figure 3.14, where the controllers \( \tilde{A} \) and \( \tilde{B} \) are in the forms of (3.34) and (3.35), respectively. If the following condition is satisfied:

\[
(8 - 4\sqrt{3})m < n < (8 + 4\sqrt{3})m
\]

then the sufficient condition

\[
\left\| \tilde{A}(N + \Delta N) - \tilde{A}N + \tilde{B}(\tilde{D} + \Delta \tilde{D}) - \tilde{B}\tilde{D}M^{-1} \right\| < 1
\]

(3.38)

can be satisfied.

**Proof**: According to the obtained results, the following equality can be obtained:

\[
H(r) = [\tilde{A}(N + \Delta N) - \tilde{A}N + \tilde{B}(\tilde{D} + \Delta \tilde{D}) - \tilde{B}\tilde{D}M^{-1}(r)
\]

\[
= [(N + \Delta N) - M]M^{-1}(r) = (N + \Delta N)M^{-1}(r) - I(r)
\]

\[
= (N + \Delta N)(\tilde{A}N + \tilde{B}\tilde{D})^{-1}(r) - I(r)
\]

\[
= (N + \Delta N) \left[ \frac{m}{n} N + \frac{n-m}{n}(N + \Delta N)^2 \tilde{D} \right]^{-1}(r) - I(r)
\]

\[
= (N + \Delta N) \left[ \frac{m}{n} N^2 + \frac{n-m}{n}(N + \Delta N)^2 \right]^{-1}(r) - I(r)
\]

\[
= \left\{ \left[ \frac{m}{n} N^2 + \frac{n-m}{n}(N + \Delta N)^2 \right]N^{-1}(N + \Delta N)^{-1} \right\}^{-1}(r) - I(r)
\]

\[
= \left[ \frac{m}{n} N(N + \Delta N)^{-1} + \frac{n-m}{n}(N + \Delta N)^2 N^{-1}(N + \Delta N)^{-1} \right]^{-1}(r) - I(r)
\]

\[
= R^{-1}(r) - I(r)
\]

(3.39)
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where \( R = \frac{m}{n} N(N+\Delta N)^{-1} + \frac{n-m}{n} (N+\Delta N)^2 N^{-1}(N+\Delta N)^{-1} \), it can be found for an element \( \delta \), both \( N(N+\Delta N)^{-1}(\delta) \) and \((N+\Delta N)^2 N^{-1}(N+\Delta N)^{-1}(\delta)\) are in the linear forms of \( \delta \), so \( R(\delta) \) is invertible and the coefficient of its inverse is the multiplicative inverse of its. Moreover, according to the fact that for any two positive numbers, their geometric mean is always smaller than or equal to their arithmetic mean, the following relationship of the coefficient can be obtained:

\[
\alpha(R) = \alpha\left(\frac{m}{n} N(N+\Delta N)^{-1}\right) + \alpha\left(\frac{n-m}{n} (N+\Delta N)^2 N^{-1}(N+\Delta N)^{-1}\right) \\
\geq 2 \sqrt{\frac{m(n-m)}{n^2}} \frac{\Psi(t) + \Delta(t)}{\Psi(t)} \sqrt{r-1} \\
\geq 2 \sqrt{\frac{m(n-m)}{n^2}} \tag{3.40}
\]

According to the condition (3.37), we get \( n^2 - 16mn + 16m^2 < 0 \), which means that \( 2 \sqrt{\frac{m(n-m)}{n^2}} > \frac{1}{2} \), then the coefficient of \( R^{-1} \): \( 0 < \alpha(R^{-1}) < \frac{1}{2\sqrt{\frac{m(n-m)}{n^2}}} < 2 \), therefore

\[
\left\| [\hat{A}(N+\Delta N) - \hat{A}N + \hat{B}(\hat{D}+\Delta\hat{D}) - \hat{B}\hat{D}] M^{-1} \right\| \\
= \sup_{t \in [0, \infty)} \sup_{r_1(t), r_2(t) \in \mathbb{D}^e \atop r_1(t) \neq r_2(t)} \left\| H(r_1(t)) - H(r_2(t)) \right\| \tilde{y}_B \\
= \sup_{t \in [0, \infty)} \sup_{r_1(t), r_2(t) \in \mathbb{D}^e \atop r_1(t) \neq r_2(t)} \left\| R^{-1} [r_1(t) - r_2(t)] - I(r_1(t) - r_2(t)) \right\| \tilde{y}_B \\
= \sup_{t \in [0, \infty)} \left\| \alpha(R^{-1}) - 1 \right\| < 1
\]

**Theorem 3.10:** Considering the nonlinear feedback control system shown in Figure 3.14, where the system is well-posed and its equivalent system based on
Bezout identity is shown in Figure 3.15, by the design scheme in Theorem 3.7, the plant output tracks to the reference input.

Remark: Since in Theorem 3.7, two controllers \( \hat{A} \) and \( \hat{B} \) are proved to be stable, in Theorem 3.9, the existence domains of \( \hat{A} \) and \( \hat{B} \) can be obtained, then according to the definition of coprimeness, the right factorization of the plant realized by the isomorphism approach is coprime. Moreover, the robust stability of the nonlinear feedback control system is also guaranteed, thus, the right coprime factorization is exactly to be a robust right coprime factorization of the perturbed plant. That extends the existing results.

3.3.6 Numerical example

In this section, a numerical example is given to show the effectiveness of the proposed method, where the input space and the output space of the given nonlinear plant are denoted to be \( U \ Y \), respectively.

Consider the system in Figure 3.11 in which the input space, output space is \( U \ Y \), respectively. The given plant operator \( P : U \to Y \) is defined by follows, where \( |\lambda(t)| < 1 \),

\[
(P + \Delta P)(u)(t) = \frac{1}{15} (e^{2t} + (1 + \lambda(t))e^{t} + \lambda(t) + (\lambda(t) + \lambda^2(t))e^{-t})u^2(t)
\]

Next, based on the construct method of the isomorphic subspace and according to the characteristics of isomorphism, the following relationship reads, where \( \phi : \tilde{U} \to \tilde{W} \) is an isomorphism:

\[
\phi(w \circ u) = \phi(w) \ast \phi(u)
\] (3.41)

where \( w, u \in \tilde{U} \) and \( \ast \) is the operation defined in operator theory, thus \( w = \phi(u) \), then:

\[
\phi(\phi(u) \circ u) = \phi(\phi(u))
\] (3.42)
where the mapping “◦” is defined by:

\[ \phi(u)(t) \circ u(t) = - \frac{1}{15} u^2(t) \left( \int_0^t 15 \phi(u)(\tau) - u^2(\tau) \frac{d\tau}{u^2(\tau)} - 2 \right) \]

then by the injective property of isomorphism: \( \phi(u)(t) \circ u(t) = \phi(u)(t) \), therefore,

\[- \frac{1}{15} u^2(t) \left( \int_0^t 15 \phi(u)(\tau) - u^2(\tau) \frac{d\tau}{u^2(\tau)} - 2 \right) = \phi(u)(t) \] (3.43)

the solution of the equation (3.43) is \( \phi(u)(t) = \frac{1}{15}(1 + e^{-t})u^2(t) \), so we can define the operators \( N + \Delta N \) and \( (\tilde{D} + \Delta \tilde{D})^{-1} \) to be:

\[ (N + \Delta N)(w)(t) = \frac{e^t + 1 + \lambda(t)}{15e^t} w^2(t) \]

\[ (\tilde{D} + \Delta \tilde{D})^{-1}(u)(t) = \frac{e^t + 1 + \lambda(t)}{15e^t} u^2(t) \] (3.44)

then according to the definition of right factorization:

\[ (D + \Delta D)^{-1}(u)(t) = \sqrt{e^{2t} + \lambda(t)} u(t) \] (3.45)

From Figure 3.12 and Figure 3.13, we can find the stable \( S^{-1} \),

\[ S^{-1}(z)(t) = \frac{15e^t}{4(e^t + 1 + \lambda(t)) \sqrt{e^{2t} + \lambda(t)}} \left( \sqrt{1 - \frac{4(e^t + 1 + \lambda(t))}{15e^t} z(t) - 1} \right)^2 \] (3.46)

Next, the controllers \( \tilde{A} \) and \( \tilde{B} \) can be designed as follows according to Theorem 3.7 and Theorem 3.9:

\[ \tilde{A}(y)(t) = \frac{1}{2} y(t) \]

\[ \tilde{B}(u)(t) = \frac{1}{2} \left( \frac{e^t + 1 + \lambda(t)}{15e^t} \right)^3 u^4(t) \] (3.47)
in the following parts, we verify the sufficient conditions of the robust stability.

\[(AN + \tilde{B}\tilde{D})(w)(t) = \frac{e^t + 1}{30e^t}w^2(t) + \frac{(e^t + 1 + \lambda(t))^3}{30e^t(e^t + 1)^2}w^2(t) = M(w)(t)\] (3.48)

then the inverse of \(M\) is

\[M^{-1}(w)(t) = \sqrt{\frac{30e^t(e^t + 1)^2}{(e^t + 1)^3 + (e^t + 1 + \lambda(t))^3}}w(t) \] (3.49)

\(M\) and \(M^{-1}\) can be proved to be stable, therefore, the operator \(M\) is a unimodular operator.

Moreover, the following two conditions are also satisfied:

\[\left[\tilde{A}(N + \Delta N) + \tilde{B}(\tilde{D} + \Delta\tilde{D})\right](w)(t) = \tilde{A}\left(\frac{e^t + 1 + \lambda(t)}{15e^t}w^2(t)\right) + \tilde{B}\left(\sqrt{\frac{15e^t}{e^t + 1 + \lambda(t)}}w(t)\right)\]

\[= \frac{1}{2} \left(\frac{e^t + 1 + \lambda(t)}{15e^t}w^2(t)\right) + \frac{1}{2} \left(\frac{e^t + 1 + \lambda(t)}{15e^t}\right)^3 \left(\sqrt{\frac{15e^t}{e^t + 1 + \lambda(t)}}w(t)\right)^4\]

\[= \frac{e^t + 1 + \lambda(t)}{15e^t}w^2(t) = (N + \Delta N)(w)(t) = \tilde{M}(w)(t)\]

\[\|H\| = \left\|\tilde{A}(N + \Delta N) - \tilde{A}N + \tilde{B}(\tilde{D} + \Delta\tilde{D}) - \tilde{B}\tilde{D}\right\|_{M^{-1}}\]

\[= \sup_{t \in [0, \infty)} \sup_{r_1(t), r_2(t) \in D^p} \left\|H(r_1(t)) - H(r_2(t))\right\|_{Y^e} \left\|r_1(t) - r_2(t)\right\|_{Y^e}\]

\[= \sup_{t \in [0, \infty)} \left\|\frac{2(e^t + 1)^2(e^t + 1 + \Delta(t))}{(e^t + 1)^3 + (e^t + 1 + \Delta(t))^3} - 1\right\|\]

\[= \sup_{t \in [0, \infty)} \left\|\frac{[e^{2t} + (2 + 3\Delta(t))e^t + (1 + 3\Delta(t) + \Delta^2(t))\Delta(t)]}{(e^t + 1)^3 + (e^t + 1 + \Delta(t))^3}\right\|\]

\[= \left\|\frac{(\lambda^2(t) + 6\lambda(t) + 4)\lambda(t)}{8 + (2 + \lambda(t))^3}\right\| < 1\]
then the nonlinear feedback control system with perturbations can be found to be robust stable, and the output tracking property can be also guaranteed. From Figure 3.15, we can find that:

\[ y(t) = (N + \Delta N) \tilde{M}^{-1}(r)(t) = r(t) \]  

(3.50)

Since (3.48) is satisfied by the two stable controllers \( \tilde{A} \) and \( \tilde{B} \), according to the definition of right coprime factorization, the right factorization of the perturbed plant is coprime. In addition, the nonlinear feedback control system is robust stable, so the right coprime factorization is exactly to be robust right coprime factorization.

The simulation results are shown in Figures 3.16 and 3.17, it is easy to find that the plant output tracks to the reference input while the reference input, quasi-state, and plant output are all stable, where the reference input is \( r^*(t) = 0.01(1 + e^{-t}) \), \( \lambda(t) = e^{-t} \).
3.4 Conclusion

In this chapter, the robust right coprime factorization is considered by using isomorphism which enriches operator based theory. Firstly, the right factorization of the given nonlinear plant is realized by using isomorphism, then the existence of the two stable controllers is discussed by which the right factorization is proved to be coprime, namely, the right coprime factorization is realized. Secondly, the system design problem is discussed, in detail, after realizing the right factorization of the perturbed nonlinear plant, two stable controllers are designed from their existence domains to satisfy the robust conditions, therefore, the robust stability of the nonlinear feedback system is guaranteed, which also means that the right factorization of the nonlinear perturbed plant is a robust right coprime factorization. Meanwhile, the universal condition is also satisfied and then the plant output tracks to the reference input. The effectiveness of the proposed methods is confirmed by the simulation results.
Chapter 4

Operator based robust control of nonlinear systems

4.1 Introduction

In Chapter 3, the right coprime factorization of the given nonlinear plant is discussed and solved by using isomorphism. However, uncertainties and perturbations exist in most of the real systems which usually affect the stability and security of the nonlinear systems. Therefore, the robust control is critical which is to find a stabilizing controller to stabilize the nonlinear system with nominal plant as well as that with perturbed plant if the perturbations are bounded. There are many methods of dealing with the robust issues in many fields and many areas, such as linear matrix inequality method (LMI), sliding mode control method (SMC), robust right coprime factorization method (RRCF) and so on. Among these methods, the operator based robust right coprime factorization method is effective to deal with the robust issues and it has been proved to be a promising method in the control and design of the nonlinear systems with perturbations. Therefore, in this chapter, the operator based robust control for the nonlinear systems with perturbations will be further considered.
In Section 4.2, the isomorphism based factorization method is further considered, in details, the isomorphic subspace of the input space is logically constructed using Lipschitz norm described conditions, which enriches and extends the application of the isomorphism based right coprime factorization method. Moreover, a quantitative robust control scheme is proposed to guarantee that the robust stability of the nonlinear feedback control system and the plant output tracks to the reference input. The validity of the design scheme is confirmed by the simulation results.

In Section 4.3, considering the unknown perturbations, a design scheme of robust controllers is proposed to simplify the satisfaction of the nominal Bezout identity and perturbed Bezout identity, namely, the stability of the nominal nonlinear system and the perturbed nonlinear systems can be guaranteed. Meanwhile, the plant output asymptotically tracks to the reference input. A numerical example is given to illustrate the validity of the proposed robust control scheme.

In Section 4.4, the operator based robust right coprime factorization and robust control on the nonlinear feedback systems are summarized.

4.2 Realization of robust right coprime factorization using Lipschitz norm described conditions

4.2.1 Problem statement

The right coprime factorization of the given plant is realized by using isomorphism approach, but in the process of constructing a stable isomorphic subspace of the input space, the compensator is designed case by case, therefore, the isomorphism based right coprime factorization method will be generalized
by using Lipschitz norm described conditions in this section.

First, a sufficient condition of stabilizing a feedback system in [57] is introduced.

Consider the feedback system shown in Figure 4.1, and let $X^e$ be the extended linear space associated with a given Banach space $X_B$, to which $u^*$, $e^*$ and $y^*$ belong, and $U^*$ and $Y^*$ are denoted as follows:

$$U^* \subset X^e \text{ where } U^{e*} = \{ u^* : \|u^*_T\|_{X^e} < \infty \text{ for all } T < \infty \}$$

$$Y^* \subset X^e \text{ where } Y^{e*} = \{ y^* : \|y^*_T\|_{X^e} < \infty \text{ for all } T < \infty \}$$

Moreover, let

$$D^* = \{ e^* : e^* = u^* - y^*, u^* \in U^*, y^* \in Y^* \}$$

$\Pi^*$ is defined to be a subset of $\text{Lip}(D^*)$ such that

$$\Pi^* = \{ C^* \in \text{Lip}(D^*) : P^*C^* \in \text{Lip}(D^*) \} \tag{4.1}$$

Lemma 4.1 Let $\Pi^*$ be the admissible class of nonlinear compensator $C^*$ defined by (4.1) for the feedback system shown in Figure 4.1, then the subset $\Pi_0^*$ of $\Pi^*$ defined below consists of the compensators $C^*$ that stabilize the overall feedback systems:

$$\Pi_0^* = \{ C^* \in \Pi^*, \|P^*C^*\| < 1 \} \tag{4.2}$$
where $\| \cdot \|$ is the Lipschitz norm defined in (2.7).

**Proof.** The proof is given in Appendix A.7 [57].

### 4.2.2 Lipschitz norm described conditions

Based on **Lemma 4.1**, the following theorem can be obtained.

**Theorem 4.1** Suppose that $\Omega = \{ z : z = u - K(w), u \in U, w \in \tilde{W} \}$, and let $\Sigma = \{ S^{-1} \in \text{Lip}(\Omega) : D^{-1}S^{-1} \in \text{Lip}(\Omega) \}$ be the admissible class of nonlinear series compensator $S^{-1}$ for the feedback system shown in Figure 4.2, and the feedback compensator $K$ is chosen from admissible class $\Xi = \{ K \in \text{Lip}(\Omega) : KD^{-1} \in \text{Lip}(\Omega) \}$, then the following class consists of the compensators $S^{-1}$ and $K$ that stabilize the overall feedback systems:

$$\Phi = \{ S^{-1} \in \Sigma, K \in \Xi, \| KD^{-1}S^{-1} \| < 1 \} \quad (4.3)$$

**Proof:** From Figure 4.2, we can find that

$$z + KD^{-1}S^{-1}(z) = u \quad (4.4)$$

which is a vector-valued nonlinear equation in the error signal $z$ for each fixed input signal $u$. The condition (4.3) means that the composite operator $KD^{-1}S^{-1}$ is Lipschitz with a norm strictly less than 1 uniformly on bounded
subsets of $\Omega$, then by the contraction mapping theorem in [57], we can find that for each $u \in U$, there is a unique $z \in U$ to satisfy the above equation. This implies that the overall feedback system shown in Figure 4.2 is BIBO stable. This completes the proof.

Next, a control scheme of constructing a stable isomorphic subspace of the input space is given. For simplification, the control scheme is denoted to be Lipschitz norm described conditions.

**Theorem 4.2** Consider constructing a stable isomorphic subspace of the input space to realize the factorization of the plant shown in Figure 4.3. If the two compensators are designed to satisfy the following conditions, respectively,

$$SD + K = \tilde{D}$$

$$\|[D^{-1}S^{-1}(z_1)]_T - [D^{-1}S^{-1}(z_2)]_T\|_{W^c} \leq L_1\|[z_1]_T - [z_2]_T\|_{U^c}$$

$$\|[K(w_1)]_T - [K(w_2)]_T\|_{U^c} \leq L_2\|[w_1]_T - [w_2]_T\|_{\tilde{W}^c}$$

where $L_1$ and $L_2$ are constants with $L_2L_1 < 1$, then the obtained isomorphic subspace $\tilde{W}$ of input space $U$ is a space with BIBO stable operators. Proof:

![Figure 4.3: Obtained two stable factors](image)

It can be seen from Figure 4.2 and Figure 4.3, if the two compensators are
designed to satisfy (4.5), then the system shown in Figure 4.2 is consequently equivalent to the part $\hat{D}^{-1}$ shown in Figure 4.3.

Moreover, we can find that the following conditions are satisfied for $T \in [0, \infty)$ and $z_1, z_2 \in U^e, [z_1]_T \neq [z_2]_T$:

$$
\frac{[K(w_1)]_T - [K(w_2)]_T}{U^e} \\
\leq L_2\|[w_1]_T - [w_2]_T\|_{W^e} \\
\leq L_2\|[D^{-1}S^{-1}(z_1)]_T - [D^{-1}S^{-1}(z_2)]_T\|_{W^e} \\
\leq L_2L_1\|[z_1]_T - [z_2]_T\|_{U^e}
$$

That means,

$$
\frac{[KD^{-1}S^{-1}(z_1)]_T - [KD^{-1}S^{-1}(z_2)]_T}{U^e} \\
= \frac{[K(w_1)]_T - [K(w_2)]_T}{U^e} \\
\leq L_2L_1\|[z_1]_T - [z_2]_T\|_{U^e}
$$

Therefore, we can find that

$$
\frac{[KD^{-1}S^{-1}]}{U^e} \leq L_2L_1 < 1
$$

Then according to Theorem 4.1, the feedback system shown in Figure 4.2 is BIBO stable, which means that the designed series compensator $S^{-1}$ and feedback compensator $K$ can stabilize the unstable part $D^{-1}$. This completes the proof.

In this part, Theorem 4.1 is concerned with a control scheme of stabilizing a feedback system with a series compensator and a feedback compensator, moreover, some sufficient conditions of designing the two compensators are obtained in Theorem 4.2.

Two points worth mentioning: one is, the design of two compensators ensures the BIBO stability of the operators in the constructing isomorphic
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subspace by satisfying the sufficient conditions in \textbf{Theorem 4.2}. Moreover, the constructed feedback system includes the feedback system with unity feedback. The other is, the design of two compensators generalizes the construction of the isomorphic subspace of the input space.

Therefore, based on the sufficient conditions of \textbf{Theorem 4.2}, the stable isomorphic subspace of the input space can be logically constructed and then the right factorization of the nonlinear plant can be realized.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig4.4}
\caption{Nonlinear feedback system obtained by isomorphism}
\end{figure}

\textbf{Numerical example}

Consider the system in Figure 4.4 in which the input space, output space is \( U, Y \), respectively. The real plant and the model plant are respectively given as

\[
\hat{P}(u)(t) = (P + \Delta P)(u)(t) = \frac{1}{3}[e^{2u} + (1 + \eta_1)]u^2(t)
\]

\[
P(u)(t) = \frac{1}{3}[e^{2u} + e^l]u^2(t)
\]

where \( \eta_1 \) is bounded with \(|\eta_1| < 3\). The given plants are known to be able to factorize into two parts as follows:

\[
\hat{P}(u)(t) = (P + \Delta P)(u)(t) = (N + \Delta N)D^{-1}(u)(t)
\]

\[
P(u)(t) = ND^{-1}(u)(t)
\]
where nonlinear unstable part is included in $D^{-1}$ whereas $N$ and $N + \Delta N$ contain the linear stable parts.

To realize the factorization of the real plant and the model plant by using isomorphism, an isomorphic subspace of the input space is constructed by a similar nonlinear feedback system shown in Figure 4.2 where the series compensator $S^{-1}(z)(t) = \sqrt{\frac{3e^{-2t}(1+e^{-t})}{8-e^{-t}}}z(t)$ and the feedback compensator $K(w)(t) = \frac{1}{3}w(t)$ are designed according to Theorem 4.2. Therefore, the isomorphic subspace of the input space is constructed and the mapping $\phi: U \rightarrow \tilde{W}$ is an isomorphism between the two spaces, where ‘$\circ$’ is defined by:

$$\phi(w \circ u)(t) = -\frac{1}{3}u(t)(\int_0^t \frac{3w(\tau) - u(\tau)}{u(\tau)}d\tau - 2) \quad (4.11)$$

then by the definition of isomorphism

$$\phi(w \circ u) = \phi(w) * \phi(u) \quad (4.12)$$

where $w, u \in U$ and “$*$” is the operation in operator theory, therefore $w = \phi(u)$ is established, then:

$$\phi(\phi(u) \circ u) = \phi(\phi(u)) \quad (4.13)$$

then by the injective property of isomorphism:

$$\phi(u)(t) \circ u(t) = \phi(u)(t) \quad (4.14)$$

therefore from (4.11) and (4.14)

$$\phi(u)(t) = \frac{1}{3}(1 + e^{-t})u(t) \quad (4.15)$$

So the corresponding operators are obtained as follows:

$$N(w)(t) = \frac{1}{3}(1 + e^{-t})w(t)$$

$$(N + \Delta N)(w)(t) = \frac{1}{3}(1 + e^{-t} + \eta_1 e^{-t})w(t)$$

$$\tilde{D}^{-1}(u)(t) = \frac{1}{3}(1 + e^{-t})u(t)$$

$$D^{-1}(u)(t) = e^{2t}u^2(t) \quad (4.16)$$
Therefore, the factorization of the plant has been realized. Next two controllers will be designed to guarantee the robust stability of the whole nonlinear feedback control system.

Based on the sufficient conditions in Section 3.2, the two controllers can be designed as follows for the given unimodular $M(w)(t) = (\frac{1}{3} + \frac{1}{6}e^{-t})w(t)$.

$$\tilde{A}(y)(t) = \frac{1}{2} y(t)$$
$$\tilde{B}(u)(t) = \frac{1}{18} (1 + e^{-t})u(t)$$

It is obvious that the following equation is satisfied:

$$(\tilde{A}N + \tilde{B}D)(w)(t) = \frac{1}{6}(1 + e^{-t})w(t) + \frac{1}{18}(1 + e^{-t})(\frac{3}{1 + e^{-t}})w(t)$$
$$= (\frac{1}{3} + \frac{1}{6}e^{-t})w(t) = M(w)(t)$$ (4.17)

Therefore, the right factorization is a right coprime factorization of the nonlinear plant.

Moreover, the following condition is also satisfied:

$$(\tilde{A}(N + \Delta N) + \tilde{B}D)(w)$$
$$= \frac{1}{6}(1 + e^{-t} + \eta_1 e^{-t})(w) + \frac{1}{18}(1 + e^{-t})(\frac{3}{1 + e^{-t}})(w)$$
$$= (\frac{1}{3} + \frac{1}{6}(1 + \eta_1)e^{-t})(w)$$
$$= \tilde{M}(w)$$

By the definition of unimodular, the operator $\tilde{M}$ is found to be a unimodular operator.

Further, the robust condition is verified as follows, where $H(r) = [\tilde{A}(N +
\[ \Delta N) - \tilde{A}N]M^{-1}(r) \]

\[
\|[[\tilde{A}(N + \Delta N) - \tilde{A}N]M^{-1}] \|
\]

\[
= \sup_{T \in [0, \infty)} \sup_{r_1, r_2 \in D} \sup_{r_1 T \neq r_2 T} \|H(r_1)T - [H(r_2)]T\|Y
\]

\[
= \sup_{T \in [0, \infty)} \left\| \frac{\eta_1}{2e^t + 1} \right\|
\]

\[
= \frac{1}{3} |\eta_1| < 1
\]

Therefore, according to Lemma 2.6, we can find that the whole nonlinear feedback control system is robustly BIBO stable.

The simulation results are shown in Figures 4.5 - 4.7, the reference input, quasi-state and the plant output are stable, where the reference input is \( r_1(t) = 1 + e^{-t} \).

Figure 4.5: Reference input \( r_1 \)
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Figure 4.6: Plant output $y_1$

Figure 4.7: Quasi-state $w_1$
4.2.3 Quantitative robust control scheme

Since the uncertainties are almost unavoidable in the real systems and they usually affect stability of the whole systems, then in this section, the robust stability of the nonlinear feedback system and plant output tracking property will be discussed. That is, a quantitative robust control scheme is proposed to guarantee that the robust stability of the nonlinear feedback control system and the plant output tracks to the reference input.

**Theorem 4.3** Consider a nonlinear system shown in Figure 4.8, where the system is well-posed and \((N + \Delta N)\) is a unimodular operator, if \(\hat{A}\) is designed to tend to \(I\) and stable \(\hat{B}\) is designed such that \(\hat{B}D(N + \Delta N)^{-1}\) is asymptotically stable, moreover, the following conditions are satisfied:

\[
\hat{A}N + \hat{B}D = M \in \mu(W, Y) \tag{4.18}
\]

\[
\hat{A}(N + \Delta N) + \hat{B}D = N + \Delta N \tag{4.19}
\]

then the nonlinear feedback system shown in Figure 4.8 is stable and plant output tracks to the reference input.

**Proof:** First, define \(\hat{M} = N + \Delta N\), then \(\hat{M}\) is a unimodular operator by the
assumption of \( N + \Delta N \) being unimodular, then (4.19) is a Bezout identity, therefore, according to Lemma 2.5, the system shown in Figure 4.8 is BIBO stable.

Next, if \( \hat{A} \) is designed to tend to \( I \), and the stable \( \hat{B} \) is designed to satisfy the condition \( \hat{B}D(N + \Delta N)^{-1} \to 0 \), moreover, the designed \( \hat{A} \) and \( \hat{B} \) satisfy (4.19), then the plant output tracks to the reference input. This completes the proof.

**Theorem 4.3** supplies a quantitative control scheme of two stable controllers \( \hat{A} \) and \( \hat{B} \) to guarantee the sufficient conditions of robust stability and output tracking property. The two controllers \( \hat{A} \) and \( \hat{B} \) are designed by considering asymptotic stability of the error signal and the plant output tracking to the reference input property.

**Numerical example**

Consider the system in Figure 4.8 where the input space, output space is \( U, Y \). The real plant and the model plant are respectively given

\[
\hat{P}(u)(t) = (P + \Delta P)(u)(t) = (e^t + 1 + \eta_2)u^2(t) \\
P(u)(t) = (e^t + 1)u^2(t)
\]

where \( \eta_2 \) is bounded with \( |\eta_2| < 9 \).

The real plant and the model plant can be factorized as follows.

\[
N(w)(t) = (1 + e^{-t})w(t) \\
(N + \Delta N)(w)(t) = (1 + e^{-t} + \eta_2 e^{-t})w(t) \\
D^{-1}(u)(t) = e^{t}u^2(t)
\]  

(4.20)

Moreover, \( D(w)(t) = \sqrt{e^{-t}w(t)} \).
According to Theorem 4.3, the controller $\hat{A}$ is designed

$$\hat{A}(y)(t) = \frac{1}{1 + 8e^{-t}y(t)}$$

(4.21)

and the stable controller $\hat{B}$ is designed as

$$\hat{B}(u)(t) = \frac{8(1 + e^{-t} + \eta_2 e^{-t})}{1 + 8e^{-t}}u^2(t)$$

(4.22)

Further, the following condition is also satisfied:

$$(\hat{A}N + \hat{B}D)(w)(t) = \frac{1 + 9e^{-t} + 8e^{-2t} + 8\eta_2 e^{-2t}}{1 + 8e^{-t}} = M(w)(t)$$

(4.23)

By the definition of unimodular operator, the operator $M$ is found to be unimodular.

Therefore, by two designed controllers $\hat{A}$ and $\hat{B}$, the conditions in Theorem 4.3 are satisfied, then the system is stable, meanwhile, the plant output tracks to the reference input.

The simulation results are shown in Figure 4.9-4.10, where the reference input $r_2(t) = 1 + e^{-t}$ and $\eta_2 = 0.4$.

4.3 Simplified robust control for a class of nonlinear systems with unknown perturbations

In Section 3.3, the robust right coprime factorization is realized, but for the nonlinear systems with unknown perturbations, the problem of robust control hasn’t been solved. In detail, on the system design problem, two kinds of objects should be designed. One is the factors $N + \Delta N, D$ of the perturbed plant $P + \Delta P$, the other is the design of the stabilizing controllers $A, B$. The factors $(N + \Delta N, D)$ can be obtained by the factorization method proposed
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Reference input \( r \) and plant output \( y \)

![Reference input r and plant output y](image1)

**Figure 4.9:** Plant output tracks to the reference input

Quasi-state \( w \)

![Quasi-state \( w \)](image2)

**Figure 4.10:** Quasi-state \( w_2 \)
in Section 3.3 while the design of the robust controllers \((A, B)\) for the nonlinear plant with unknown perturbations have not been solved. Moreover, since there exist two controllers \(A\) and \(B\) to be designed which increase the difficulty of the design. Moreover, the plant output perfect tracking property cannot be guaranteed in the case of unknown perturbations. Therefore, in this section, a simple design of feedforward compensator \(B\) is proposed such that the Bezout identities for the nominal nonlinear system as well as the perturbed nonlinear system can be simply satisfied. Meanwhile, by the designed controllers, the asymptotic tracking property can also be guaranteed.

4.3.1 Simplified robust control for a class of nonlinear systems with unknown perturbations

Consider the nonlinear feedback system with unknown perturbations shown in Figure 2.4, where the system is assumed to be well-posed and the factors \(N, N + \Delta N\) and \(D\) are assumed to be obtained from the isomorphism factorization method as Section 3.3 showed. That is, the nonlinear parts are included in the factor \(N + \Delta N\) while the linear parts are contained in \(D^{-1}\). Also, the existence domain of the controllers \((A, B)\) are given in Section 3.3, but the design scheme is only effective for the case of nonlinear system where the perfect tracking is guaranteed. In most cases, the perfect tracking cannot be realized, especially for the case of unknown perturbations. Therefore, in the following part, we will consider a design scheme of robust controllers \((A, B)\) for the nonlinear feedback system with unknown perturbations.

**Theorem 4.4** Supposed the nonlinear feedback system with unknown perturbations shown in Figure 2.4 is well-posed and the factors \(N + \Delta N, D^{-1}\) are obtained from the isomorphism factorization method shown in Section 3.3, where the unknown perturbations are assumed to be positive and bounded. If
the controller \( A \) is designed to tend to \( I \) and the controller \( B \) is designed to be \( B(u(t)) = \frac{1}{K}u(t) \), where \( K \) is a large gain, then the designed controllers \((A, B)\) are robust stabilizing controllers for the nonlinear system shown in Figure 2.4.

**Proof:** According to the assumption, the operators are supposed to be designed as follows

\[
N(w)(t) = \phi(t)w^\alpha(t)
\]

\[
(N + \Delta N)(w)(t) = (\phi(t) + \varepsilon(t))w^\alpha(t)
\]

\[
D(w)(t) = \psi(t)w(t)
\]

\[
A(y)(t) = \eta(t)y(t)
\]

where \( \phi(t) \) and \( \psi(t) \) are BIBO stable, the unknown parameter \( \varepsilon(t) > 0, |\varepsilon(t)| < \beta \) and \( \eta(t) \) tends to 1, then the following condition is satisfied:

\[
M(w)(t) = (AN + BD)(w)(t)
\]

\[
= \eta(t)\phi(t)w^\alpha(t) + \frac{1}{K}\psi(t)w(t)
\]

\[
\rightarrow \phi(t)w^\alpha(t) + \frac{1}{K}\psi(t)w(t)
\] (4.24)

then by the assumption of \( N, D \) and the design of \( A \) and \( B \), we can find that the operator \( M \) is bounded and it does not tend to 0 which means that \( M \) is a unimodular operator, then the Bezout identity is satisfied. Therefore, the nonlinear systems with nominal plant is stabilized by the controllers \( A \) and \( B \). Then the next work is to verify whether the controllers can stabilize the perturbed nonlinear system or not.

\[
\tilde{M}(w)(t) = [AN + \Delta N + BD](w)(t)
\]

\[
= \eta(t)(\phi(t) + \varepsilon(t))w^\alpha(t) + \frac{1}{K}\psi(t)w(t)
\]

\[
\rightarrow (\phi(t) + \varepsilon(t))w^\alpha(t) + \frac{1}{K}\psi(t)w(t)
\] (4.25)
Figure 4.11: Equivalent system between two different spaces

By the assumptions and conditions, $\varepsilon(t)$ is positive and bounded while $\eta(t) \to 1$, the operator $\tilde{M}$ is proved to be a unimodular operator, then (4.25) is proved to be a Bezout identity, based on Lemma 2.5, the stability of the nonlinear system with unknown perturbations is guaranteed. Then, by the designed controllers $(A, B)$, both the nonlinear system with nominal plant and the nonlinear system with unknown perturbations are guaranteed to be stable, therefore, the designed controllers $(A, B)$ are robust stabilizing controllers. This completes the proof.

In this proof, the Bezout identities (4.24) and (4.25) are simply fulfilled by the designed robust controllers $A$ and $B$, which also means that the stability of the nonlinear feedback system shown in Figures 2.2 and 2.4 are both guaranteed.

Next, the plant output tracking to the reference input problem will be discussed. For simplification, only the case of $\alpha = 2$ is considered in the following parts, then Theorem 4.5 can be obtained for the nonlinear system shown in Figure 4.11 which is equivalent with the nonlinear feedback system shown in Figure 2.4.

**Theorem 4.5** Considering the nonlinear equivalent system shown in Figure 4.11, by the robust stabilizing controllers $(A, B)$, the plant output can be found to asymptotically track to the reference input.
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Proof: Since the operator $\tilde{M}$ is a unimodular operator, then the inverse of it is obtained by completing square of some corresponding parameters as follows:

$$\tilde{M}^{-1}(r)(t) = \left[ \frac{1}{\eta(t)(\phi(t) + \varepsilon(t))} \right]^{\frac{1}{2}} F(r)(t) \quad (4.26)$$

where

$$F(r)(t) = \frac{\sqrt{\psi^2(t) + 4r(t)K^2\eta(t)(\phi(t) + \varepsilon(t))} - \psi(t)}{2K\sqrt{\eta(t)(\phi(t) + \varepsilon(t))}} \quad (4.27)$$

then based on (4.26) and (4.27), the following condition is satisfied

$$(N + \Delta N)\tilde{M}^{-1}(r)(t)$$

$$= [\phi(t) + \varepsilon(t)](\frac{1}{\eta(t)(\phi(t) + \varepsilon(t))})F^2(r)(t)$$

$$= \frac{1}{\eta(t)} \left[ r(t) - \frac{\psi(t)[\sqrt{\psi^2(t) + 4r(t)K^2\eta(t)(\phi(t) + \varepsilon(t))} - \psi(t)]}{2K^2\eta(t)(\phi(t) + \varepsilon(t))} \right] \quad (4.28)$$

since $K$ is large enough, then (4.28) tends to $\frac{1}{\eta(t)}r(t)$, moreover, by the assumption, $\frac{1}{\eta(t)} \to 1$, the following condition is satisfied:

$$y(t) = (N + \Delta N)\tilde{M}^{-1}(r)(t) \to r(t) \quad (4.29)$$

therefore, the plant output asymptotically tracks to the reference input. This completes the proof.

In this part, the design of the robust stabilizing controllers $A$ and $B$ are given in Theorem 4.4, wherein the controller $A$ is designed to tend to $I$ and $B$ is designed to be in a form of low gain. In detail, for the general case that the plant output cannot track to the reference input, robust stabilizing controllers $(A, B)$ are designed so that the Bezout identities can be simply satisfied, therefore, the stability of the nominal nonlinear system and perturbed nonlinear system are guaranteed. Moreover, by the robust stabilizing controllers, the plant output asymptotically tracks the reference input which is shown in Theorem 4.5.
4.3.2 Numerical example

In this part, a numerical example is given to confirm the validity of the proposed method.

Consider the nonlinear feedback system with unknown perturbations shown in Figure 2.4, by using isomorphism factorization method (Section 3.3), the nominal plant and real plant is respectively factorized as follows:

\[
N(w)(t) = (1 + e^{-t})w^2(t)
\]
\[
(N + \Delta N)(w)(t) = \left[1 + (1 + \varepsilon)e^{-t}\right]w^2(t)
\]
\[
D^{-1}(u)(t) = e^t u(t)
\]

where uncertain factor $\varepsilon$ is bounded with $|\varepsilon| < 1$, and the operator $D$ can be obtained as $D(w)(t) = e^{-t}w(t)$, then we can find that the factors $N$ and $N + \Delta N$ are unimodular operators, $D$ is stable and $D^{-1}$ is unstable. Next, we will design the stabilizing controllers to stabilize not only the nonlinear systems with nominal plant but also the one with unknown perturbations.

For simplification, the controllers $A$ and $B$ are respectively designed according to Theorem 4.4 as follows:

\[
A(y)(t) = \frac{1}{1 + e^{-t}y(t)}
\]
\[
B(u)(t) = \frac{1}{K}u(t)
\]

then we will verify the satisfaction of the Bezout identity for the nonlinear feedback system with nominal plant and perturbed plant, respectively.

Suppose that $M = AN + BD$, then the following is obtained:

\[
M(w)(t) = (AN + BD)(w)(t) = w^2(t) + \frac{1}{Ke^t}w(t)
\]

then we can find that $M$ is a unimodular operator.
Similarly,
\[
\tilde{M}(w)(t) = [A(N + \Delta N) + BD](w)(t) = \frac{1 + [1 + \varepsilon]e^{-t}}{e^{-t} + 1}w^2(t) + \frac{e^{-t}}{K}w(t)
\]

According to the definition of unimodular operator and the method of completing the square, we can find that \(\tilde{M}\) is unimodular with
\[
\tilde{M}^{-1}(r)(t) = \frac{\sqrt{\pi^2(t)e^{-2t} + 4r(t)\pi(t)K^2} - \pi(t)e^{-t}}{2K} \quad (4.33)
\]
where \(\pi(t) = \frac{1 + \varepsilon}{1 + (1 + \lambda(t))e^{-t}}\), then based on the equivalent system, we can find that
\[
y(t) = (N + \Delta N)\tilde{M}^{-1}(r) = \frac{e^t + 1 + \lambda(t)}{e^t} \left( \frac{\sqrt{\pi^2(t)e^{-2t} + 4r(t)\pi(t)K^2} - \pi(t)e^{-t}}{2K} \right)^2
\]
\[
= \frac{e^t + 1 + \lambda(t)}{e^t} \left( \frac{\pi^2(t)e^{-2t} + 2r(t)\pi(t)K^2 - \pi(t)e^{-t}\sqrt{\pi^2(t)e^{-2t} + 4r(t)\pi(t)K^2}}{2K^2} \right)
\]

by designing \(K\) as Theorem 4.5 showed,
\[
\frac{\pi^2(t)e^{-t} - \pi(t)e^{-t}\sqrt{\pi^2(t)e^{-2t} + 4r(t)\pi(t)K^2}}{2K^2} \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty \quad (4.34)
\]
then \(y(t) \rightarrow r(t)\), therefore, the plant output asymptotically tracks to the reference input.

The simulation results are given in Figures 4.12 - 4.14, from them, we can find that the plant output asymptotically tracks to the reference input while the quasi-state, plant output and reference input are all BIBO stable, where \(r_3(t) = 10 + e^{-t}, K = 1000, \varepsilon = 0.1\ rand\).

Actually, in many cases, the asymptotic tracking can be guaranteed in a finite time interval instead of the case of \(t \rightarrow \infty\) which is verified in the simulation results.
Figure 4.12: Plant output $y_3$ asymptotically tracks to reference input $r_3$

Figure 4.13: Control input $u_3$
4.4 Conclusion

In this chapter, the realization and control of the robust right coprime factorization of the given nonlinear plant are considered. Firstly, based on Lipschitz norm described conditions, the isomorphism based factorization is extended by logically constructing the isomorphic subspace of the input space. Moreover, for the uncertain nonlinear feedback system, a quantitative robust control scheme is proposed to guarantee the robust stability of the nonlinear feedback system while the plant output can also be guaranteed to track to the reference input. Secondly, considering the nonlinear feedback system with unknown perturbations, a robust control scheme is given to simplify the satisfaction of the Bezout identities while the plant output asymptotically tracks to the reference input. The effectiveness of the proposed methods is confirmed by the numerical examples.

Figure 4.14: Quasi-state of $w_3$
Chapter 5

Conclusions

In this dissertation, the operator based robust right coprime factorization method and robust control for the nonlinear feedback systems are discussed. Firstly, the fundamental problem—factorization method of the given nonlinear plant is solved by using isomorphism. Secondly, some robust control schemes are proposed by which both the nominal nonlinear systems and the perturbed nonlinear systems are guaranteed to be bounded input bounded output stable. Meanwhile, the desired plant output tracking property can be guaranteed.

In Chapter 2, the mathematical preliminaries and problem statement are included. Firstly, some basic definitions and notations of spaces and operators are introduced, especially the extended linear space associated with Banach space and generalized Lipschitz operator which serve the mathematical basis of this dissertation. Therein, some important properties such as the bounded input bounded output stable are also introduced. Secondly, the definitions about right coprime factorization, well-posedness, overall stable and robust right coprime factorization as well as the sufficient conditions of guaranteeing the right factorization to be robust right coprime factorization are outlined, which provides the theoretical knowledge of this research. Thirdly,
the basic mathematical concept—isomorphism as well as its characteristics is introduced which plays an important part in this research. Finally, the concerned and researched problems are stated and discussed.

In Chapter 3, the fundamental problem in the operator based theory, the method of right factorizing the nonlinear plant is discussed. At first, based on isomorphism, the right factorization of the given nonlinear unstable plant is realized and then for the obtained nonlinear systems, the design of two stable controllers are discussed, respectively. By them, the Bezout identity is satisfied which means that the right factorization is coprime. Meanwhile, the plant output tracking to the reference input property is discussed, respectively. Then, the system design problem is considered, where the existence domains of the two stable controllers are obtained, and by the designed stabilizing controllers, the robust conditions as well as the universal condition are satisfied, namely, the perturbed nonlinear systems are guaranteed to be robust bounded input bounded output stable while the plant output tracks to the reference input. Since the robustness of the factorization is linked to the robust stability of the nonlinear systems, therefore, the right factorization is exactly to be a robust right coprime factorization of the perturbed plant. The numerical examples are provided to show the effectiveness of the proposed design schemes.

In Chapter 4, the operator based robust control of the nonlinear systems as well as the plant output tracking property is concerned. First of all, based on the Lipschitz norm described conditions, the isomorphic subspace of input space is logically constructed which extends and enriches the application of the isomorphism based factorization method. Moreover, a quantitative design scheme of robust controllers on the nonlinear systems with perturbations is considered by which the universal condition is satisfied, that is, the
stability of the nonlinear system as well as the plant output tracking property are guaranteed. Finally, a robust control scheme is proposed to simplify the satisfaction of the Bezout identity for the nominal nonlinear systems and the perturbed nonlinear systems, respectively, which leads the stability of the nominal nonlinear systems and the perturbed nonlinear systems simple to be guaranteed. Numerical examples are given to illustrate the validity of the proposed design schemes.
Bibliography


Appendix A

Proof

A.1 Proof of Lemma 2.1

First, it is clear that $Lip(D^e, Y^e)$ is a normed linear space. Hence, it is sufficient to verify its completeness.

Let $Q_n$ be a Cauchy sequence in $Lip(D^e, Y^e)$ such that $\|Q_m - Q_n\| \to 0$ as $m, n \to \infty$. We need to show that $\|Q_n - Q\| \to 0$ for some $Q \in Lip(D^e, Y^e)$ as $n \to \infty$.

Let $T \in [0, \infty)$ be fixed. For any $\tilde{x} \in D^e$, by definition of the Lipschitz norm with an $x_0 \in D^e$, we have

$$
\|\left[ (Q_m - Q_n)(\tilde{x}) \right]_T - \left[ (Q_m - Q_n)(x_o) \right]_T \|_{Y^e}
\leq \left\| Q_m - Q_n \right\|_{Lip} \| \tilde{x}_T - [x_o]_T \|_{X^e}
$$

so that

$$
\left\| \left[ Q_m(\tilde{x}) \right]_T - \left[ Q_n(\tilde{x}) \right]_T \right\|_{Y^e} = \left\| \left[ (Q_m - Q_n)(\tilde{x}) \right]_T \right\|_{Y^e}
\leq \left\| \left[ (Q_m - Q_n)(x_0) \right]_T \right\|_{Y^e} + \left\| Q_m - Q_n \right\|_{Lip} \| \tilde{x}_T - [x_o]_T \|_{X^e}
$$

Since the right hand side of the above tends to zero as $m, n \to \infty$, it follows that the sequence $\{[Q_n(\tilde{x})]_T\}$ is Cauchy in the range $Y^e$ (and in fact is
uniformly Cauchy over each bounded subset of the domain $D^e$). Hence, for each fixed $T \in [0, \infty)$, $\tilde{v}_T := \lim [Q_n(\tilde{x})]_T$ exists in the range $Y^e$ (and is uniform over bounded subsets of the domain $D^e$). Let $v$ be a function such that $v_T = \tilde{v}_T$ for all $T \in [0, \infty)$, and define a nonlinear operator $Q$ by $Q : \tilde{x} \to v$. Then, $Q$ satisfies $[Q(\tilde{x})]_T = \tilde{v}_T$ for all $T \in [0, \infty)$. We will show that $Q \in Lip(D^e, Y^e)$. We first note that the operator $Q$ so defined has domain $D^e$ since in the above $\tilde{x} \in D^e$ is arbitrary. We then observe that $Q$ is actually independent of $T$. Then, since $\|Q_m - Q_n\| \to 0$, we have $\lim \|Q_n\|_{Lip} = c$, a constant, so that for any $\tilde{x}_1, \tilde{x}_2 \in D^e$,

$$\|Q(\tilde{x}_1)]_T - [Q(\tilde{x}_2)]_T\|_{Y^e} = \lim_{n \to \infty} \|Q_n(\tilde{x}_1)]_T - [Q_n(\tilde{x}_2)]_T\|_{Y^e} \leq c\|\tilde{x}_1]_T - [\tilde{x}_2]_T\|_{X^e}$$

(A.3)

Therefore, taking the supremum over $D^e$ and then the supremum over $[0, \infty)$ yields

$$\sup_{T \in [0, \infty)} \sup_{\tilde{x}_1, \tilde{x}_2 \in D^e, [\tilde{x}_1]_T \neq [\tilde{x}_2]_T} \frac{\|Q(\tilde{x}_1)]_T - [Q(\tilde{x}_2)]_T\|_{Y^e}}{\|\tilde{x}_1]_T - [\tilde{x}_2]_T\|_{X^e}} \leq c$$

(A.4)

which implies that $\|Q\| \leq c < \infty$, so that $Q \in Lip(D^e, Y^e)$.

We finally verify that $\|Q_n - Q\|_{Lip} \to 0$ as $n \to \infty$. Since the above also proves (letting $\tilde{x} = x_0$ therein) that $\|[Q_n(x_0)]_T - [Q(x_0)]_T\|_{Y^e} \to 0$ as $n \to \infty$ for each $T \in [0, \infty)$, for $\epsilon > 0$ we can let $N$ be such that $\|Q_m - Q_n\|_{Lip} \leq \epsilon/2$ and $\|[Q_n - Q](x_0)]_T\|_{Y} \leq \epsilon/2$ for $m, n \geq N$. Then for any given $\tilde{x}_1, \tilde{x}_2 \in D^e$,
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we have

\[
\| [(Q - Q_n)(\tilde{x}_1)]_T - [(Q - Q_n)(\tilde{x}_2)]_T \|_{Y^*} \\
= \lim_{k \to \infty} \| [(Q_k - Q_n)(\tilde{x}_1)]_T - [(Q_k - Q_n)(\tilde{x}_2)]_T \|_{Y^*} \\
\leq \lim_{k \to \infty} \| Q_k - Q_n \|_{Lip} \| [\tilde{x}_1]_T - [\tilde{x}_2]_T \|_{X^*} \\
= \epsilon/2 \| [\tilde{x}_1]_T - [\tilde{x}_2]_T \|_{X^*}\]  

(A.5)

so that \( \| Q_n - Q \|_{Lip} \leq \epsilon \) for \( n \geq N \). This shows that \( \| Q_n - Q \|_{Lip} \to 0 \) as \( n \to \infty \) and completes the proof of the lemma.

### A.2 Proof of Lemma 2.2

Suppose that \( Q : X^e \to X^e \) is causal. Then by definition we have that \( P_T Q P_T = P_T Q \), so that if \( x_T = y_T \), then

\[
[Q(x)]_T = P_T Q(x) = P_T Q P_T(x) = P_T Q(x_T) = P_T Q(y_T) \\
= P_T Q P_T(y) = P_T Q(y) = [Q(y)]_T. \]  

(A.6)

Conversely, suppose that \( x_T = y_T \) implies \( [Q(x)]_T = [Q(y)]_T \) for all \( x, y \in X^e \) and all \( T \in [0, \infty) \). Fix a \( T \in [0, \infty) \), for any \( x \in X^e \), let \( y = x_T \), then \( x_T = y_T \), so that \( [Q(x)]_T = [Q(y)]_T \). Consequently, we have that

\[
P_T Q P_T(x) = P_T Q(x_T) = P_T Q(y) \\
= [Q(y)]_T = [Q(x)]_T = P_T Q(x). \]  

(A.7)

Since \( x \in X^e \) and \( T \in [0, \infty) \) are arbitrary, it follows that \( P_T Q P_T = P_T Q \) for all \( T \in [0, \infty) \), which implies that \( Q \) is causal.
A.3 Proof of Lemma 2.3

Since
\[
\|[Q(x)]_T - [Q(y)]_T\|_{X^e} \leq \|Q\|_{Lip}\|x_T - y_T\|_{X^e}
\] (A.8)
for all \(x, y \in X^e\) and all \(T \in [0, \infty)\). Hence, \(x_T = y_T\) implies that \([Q(x)]_T = [Q(y)]_T\) for all \(x, y \in X^e\) and all \(T \in [0, \infty)\).

A.4 Proof of Lemma 2.5

Sufficiency: Since \(M \in \mu(W, U)\), for any \(r \in U_s\), we have \(r = (AN + BD)(w)\), that is \(w = M^{-1}r \in W_s\). Moreover, since \(y = N(w), e = BD(w),\) and \(b = A(y) = AN(w)\), the stability of \(A, B, N\) and \(D\) implies that \(y \in Y_s, e \in U_s\) and \(b \in U_s\). Thus, the system is overall stable.

Necessity: First, it follows from the well-posedness and through the path of \(N\) and \(A\) that \(M : W \to U\) is invertible. Then, it can be verified that both \(M\) and \(M^{-1}\) are stable. As a result, \(M \in \mu(W, U)\).

A.5 Proof of Lemma 2.6

To begin with, we recall a sufficient condition for judging operator’s invertibility.

**Lemma A.** Let \(X_B\) and \(Y_B\) be Banach space, \(S \in Lip(X_B, Y_B)\) is an invertible operator, and \(R\) is an operator in \(Lip(X_B, Y_B)\) such that \(\|RS^{-1}\| < 1\) where, \(Lip(X_B, Y_B) = \{S: X_B \to Y_B, \|S\|_{Lip} < \infty\}\). Then, the operator \(R + S\) is invertible in \(Lip(X_B, Y_B)\) and
\[
\|(R + S)^{-1}\| \leq \|S^{-1}\|(1 - \|RS^{-1}\|)^{-1}.
\] (A.9)
**Proof of Lemma A.** We first prove the following assertion: If one operator $J \in \text{Lip}(X_B)$ such that $\|J\| < 1$, then $I - J$ is invertible and

$$\|(I - J)^{-1}\| \leq (1 - \|J\|)^{-1}. \quad (A.10)$$

In fact, for $x, y \in X_B$,

$$\|(I - J)x - (I - J)y\| \geq \|x - y\| - \|Jx - Jy\| \geq (1 - \|J\|)\|x - y\|. \quad (A.11)$$

Thus, $I - J$ is injective. Next, we show that $I - J$ is surjective and $(I - J)^{-1} \in \text{Lip}(X_B)$.

Define that $Q_0 := I$ and $Q_n = I + JQ_{n-1} \forall n = 1, 2, \cdots$, we can prove that for $x \in X_B$

$$\|Q_{n+1}(x) - Q_n(x)\| \leq \|J\|\|J(x)\|_{X_B}, \; n = 1, 2, \cdots. \quad (A.12)$$

Then for any positive integer $m$, we have that

$$\|Q_{n+m}(x) - Q_n(x)\|_{X_B} = \left\| \sum_{k=0}^{m-1} (Q_{n+k+1}(x) - Q_{n+k}(x)) \right\|_{X_B} \leq \sum_{k=0}^{m-1} \|J\|^{n+k}\|J(x)\|_{X_B} \leq \frac{\|J\|^n\|J(x)\|_{X_B}}{1 - \|J\|}. \quad (A.13)$$

Since $\|J\| < 1$ and $X_B$ is Banach space, then $Q(x) = \lim_{n \to \infty} Q_n(x)$ exists and

$$\|Q(x) - Q_n(x)\|_{X_B} = \lim_{n \to \infty} \|Q_{n+m}(x) - Q_n(x)\| \leq \frac{\|J\|^n\|J(x)\|_{X_B}}{1 - \|J\|}. \quad (A.14)$$

Since $J$ is Lipschitz and hence is continuous, the

$$Q(x) = \lim_{n \to \infty} Q_n(x) = \lim_{n \to \infty} (I + JQ_{n-1})x = x + JQx \quad (A.15)$$
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that is, \( Q = I + JQ \), namely, \((I - J)Q = I\), which implies that \( I - J \) is surjective in \( \text{Lip}(X_B) \). Then for \( r, z \in \mathcal{R}(I - L) \),

\[
\|(I - J)^{-1}r - (I - J)^{-1}z\| \leq (1 - \|J\|)^{-1}\|r - z\|. \tag{A.16}
\]

Thus, the assertion is proved. As a consequence, since \( R + S = (I + RS^{-1})S \), then \( \|RS^{-1}\| < 1 \) follows that \((I + RS^{-1})^{-1}\) exists and \((R + S)^{-1} = S^{-1}(1 + RS^{-1})^{-1}\). This completes the proof of Lemma A.

**Proof of Lemma 2.6** \( M \) is unimodular operator implies it is invertible. From

\[
AN + BD = M \tag{A.17}
\]

\[
A(N + \Delta N) + BD = \tilde{M} \tag{A.18}
\]

we have

\[
\tilde{M} = M + [A(N + \Delta N) - AN]
= [I + (A(N + \Delta N) - AN)M^{-1}]M \tag{A.19}
\]

and \((A(N + \Delta N) - AN)M^{-1} \in \text{Lip}(D^s)\), then \( I + (A(N + \Delta N) - AN)M^{-1} \) is invertible based on Lemma A, where \( I \) is the identity operator. Consequently

\[
\tilde{M}^{-1} = M^{-1}[I + (A(N + \Delta N) - AN)M^{-1}]^{-1}. \tag{A.20}
\]

Meanwhile, since \((A(N + \Delta N) - AN)M^{-1} \in \text{Lip}(D^c)\) and \( M \in \mu(W, U) \), then \( \tilde{M} \in \mu(W, U) \) provided that the system shown in Figure 2.4 is well-posed. As a result, for any \( r \in U_s, w = \tilde{M}^{-1}r \in W_s \). Further, since \( y = (N + \Delta N)(w), e = BD(w) \) and \( b = A(N + \Delta N)(w) \), the stability of \( A, B, N, D \) and \( \Delta N \) implies that \( y \in Y_s, e \in U_s \) and \( b \in U_s \). Then, the system is overall stable.
A.6 Proof of Lemma 3.1

If $M$ is unimodular operator, then $M$ is invertible. From

\[ AN + BD = M \quad (A.21) \]
\[ A(N + \Delta N) + BD = \tilde{M} \quad (A.22) \]

we have

\[ \tilde{M} = M + [A(N + \Delta N) - AN + B(D + \Delta D) - BD] \]
\[ = [I + (A(N + \Delta N) - AN + B(D + \Delta D) - BD)M^{-1}]M \quad (A.23) \]

and $(A(N + \Delta N) - AN + B(D + \Delta D) - BD)M^{-1} \in Lip(D^e)$, then $I + (A(N + \Delta N) - AN + B(D + \Delta D) - BD)M^{-1}$ is invertible, where $I$ is the identity operator. Consequently

\[ \tilde{M}^{-1} = M^{-1}[I + (A(N + \Delta N) - AN + B(D + \Delta D) - BD)M^{-1}]^{-1} \quad (A.24) \]

Meanwhile, since $(A(N + \Delta N) - AN + B(D + \Delta D) - BD)M^{-1} \in Lip(D^e)$, $M^{-1}(A(N + \Delta N) - AN + B(D + \Delta D) - BD) \in Lip(D^e)$ and $M \in \mu(W, U)$, we have $\tilde{M} \in \mu(W, U)$, that is $\tilde{M}$ is a unimodular operator. Then, the system is overall stable.

A.7 Proof of Lemma 4.1

It follows from Figure 4.1 that

\[ e^* + P^*C^*(e^*) = u^* \]

which is a vector-valued nonlinear equation in the error signal $e^*$ for each fixed input $u^*$. The condition in (4.2) ensure that the composite operator $P^*C^*$ is Lipschitz with the norm strictly less than 1 uniformly on bounded subsets
of $D^*$. By the contraction mapping theorem in [57], $P^*C^*$ is a contraction mapping from $D^*$ to itself, so that for each $u^* \in D^*$, there is a unique $e^* \in D^*$ satisfying the above equation. This implies that the overall feedback system shown in Figure 4.1 is input-output stable.
Appendix B

Publications

Journal papers


2. N. Bu and M. Deng, Isomorphism-based robust right coprime factorization realization for nonlinear feedback systems, *IMechE, Part I: Journal of Systems and Control Engineering*, vol. 225 (Accepted for publication). (Chapter 4)

Book chapter

1. N. Bu and M. Deng, Robust right coprime factorization realization for nonlinear systems using isomorphism. *Book of selected papers from International Workshop on Human Adaptive Mechatronics, Springer* (Accepted for publication). (Chapter 4)
APPENDIX B. PUBLICATIONS

Proceedings papers


2. **N. Bu** and M. Deng, Robust control for a class of nonlinear systems with unknown perturbations, *Proceedings of 2011 International Conference on Advanced Mechatronic Systems*, pp. 1-5, 2011. (Chapter 4)

Other papers


